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Slide of the Seminar

Can we quantify the effect of waves in turbulence?

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Can we quantify the effect of waves in turbulence?

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The Navier-Stokes equations

- Momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{F} \qquad \nabla \cdot \mathbf{v} = 0$$

- $\mathcal{P}$ is the pressure, $\mathbf{F}$ an external force, and ν the kinematic viscosity, incompressibility is assumed.
- Quadratic invariants ($\mathbf{F} = 0$, $\nu = 0$):

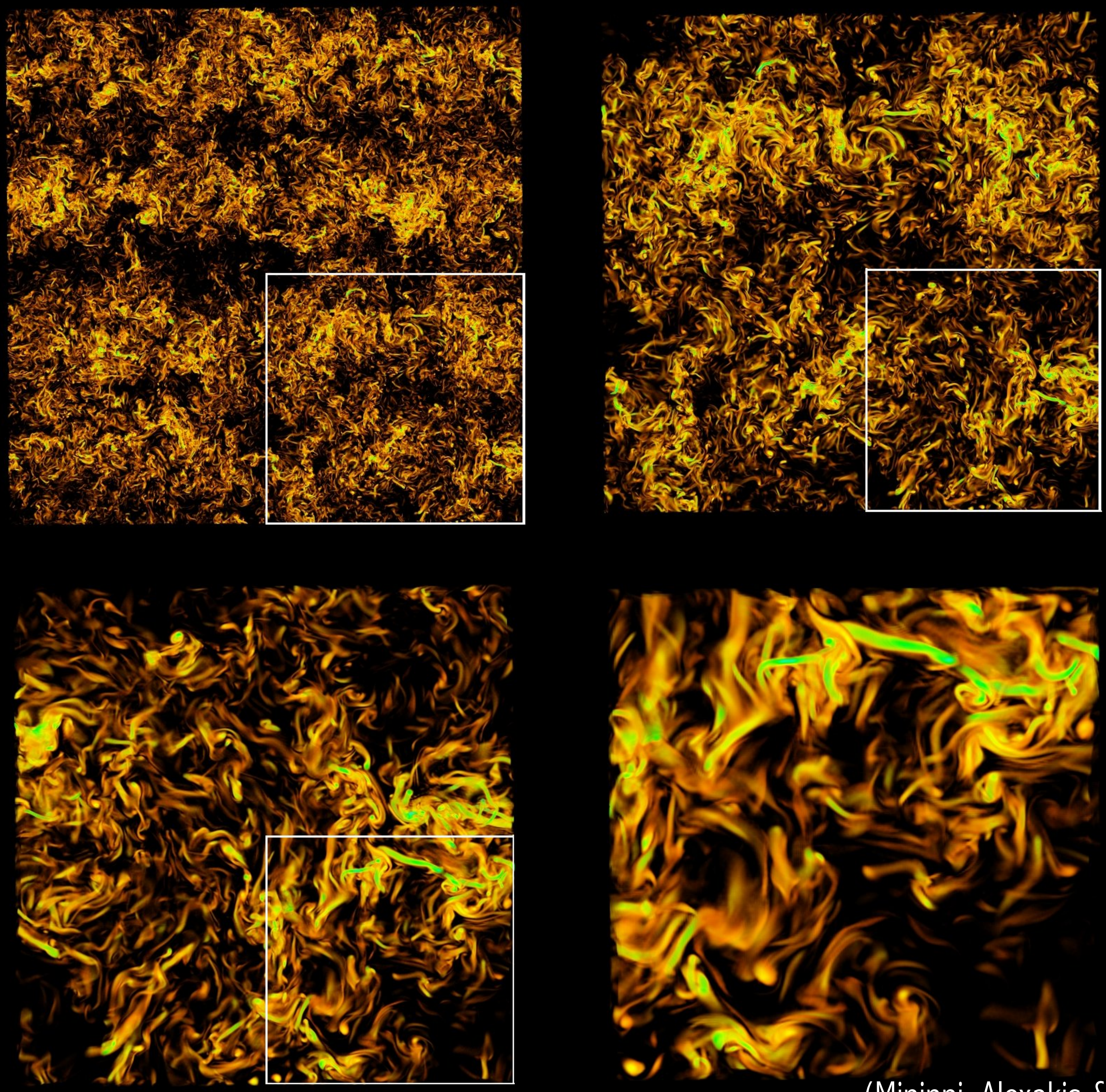
$$E = \int \mathbf{v}^2 \, \mathrm{d}^3x$$

$$H = \int \mathbf{v} \cdot \boldsymbol{\omega} \, \mathrm{d}^3x \qquad \boldsymbol{\omega} = \nabla \times \mathbf{v}$$

- Reynolds numbers:

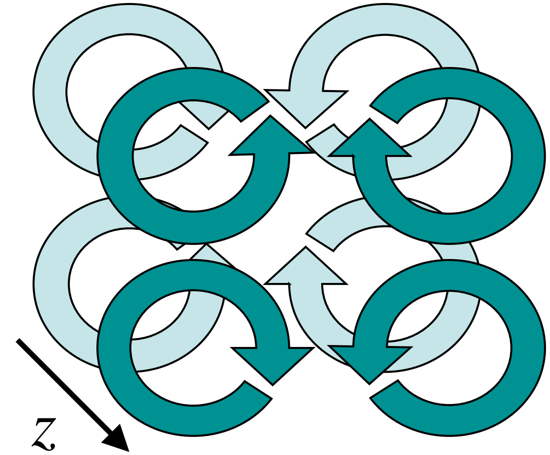
$$Re = UL / \nu \qquad R_\lambda = U\lambda / \nu$$

where L is the integral scale and λ the Taylor scale.



2048^3
(Mininni, Alexakis & Pouquet 2008)

The energy cascade

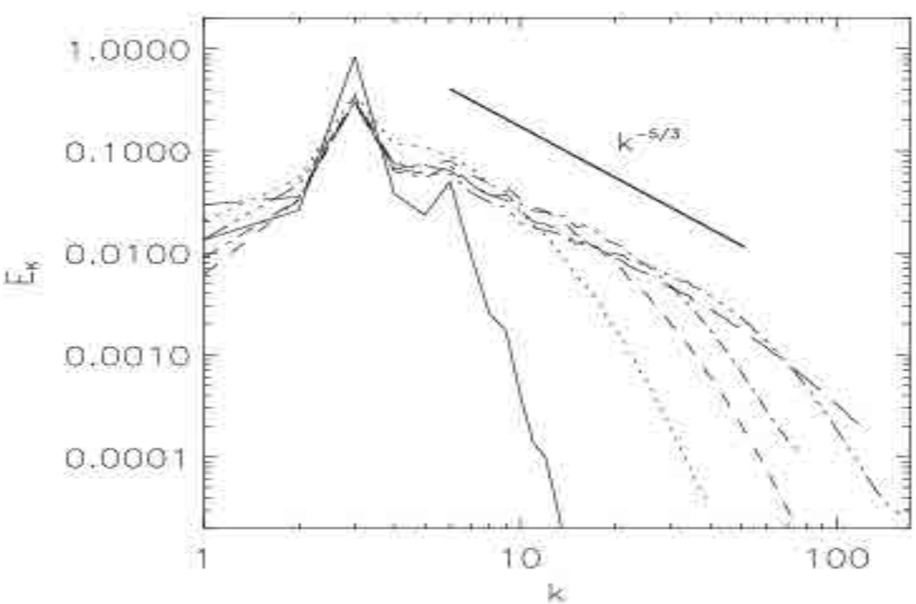


Starting from

$$\mathbf{v} = \begin{bmatrix} \sin(k_0 x)\cos(k_0 y)\cos(k_0 z) \\ -\cos(k_0 x)\sin(k_0 y)\cos(k_0 z) \\ 0 \end{bmatrix}$$

as initial condition, and replacing in the Navier-Stokes equation

$$\frac{\partial v_x}{\partial t} = \frac{k_0 \sin(2k_0 x)}{8}\left[\cos(2k_0 z) - \cos(2k_0 y)\right] - 3k_0^2 \nu \cos(k_0 x)\sin(k_0 y)\sin(k_0 z)$$



- This process can be repeated, and smaller eddies are created until reaching the scale where the dissipative term dominates! Taylor & Green, *Proc. Roy. Soc.* A **151**, 421 (1935).

Turbulence: the Navier-Stokes equations

- This leads naturally to a Fourier representation for the velocity in the momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{F} \qquad \nabla \cdot \mathbf{v} = 0$$

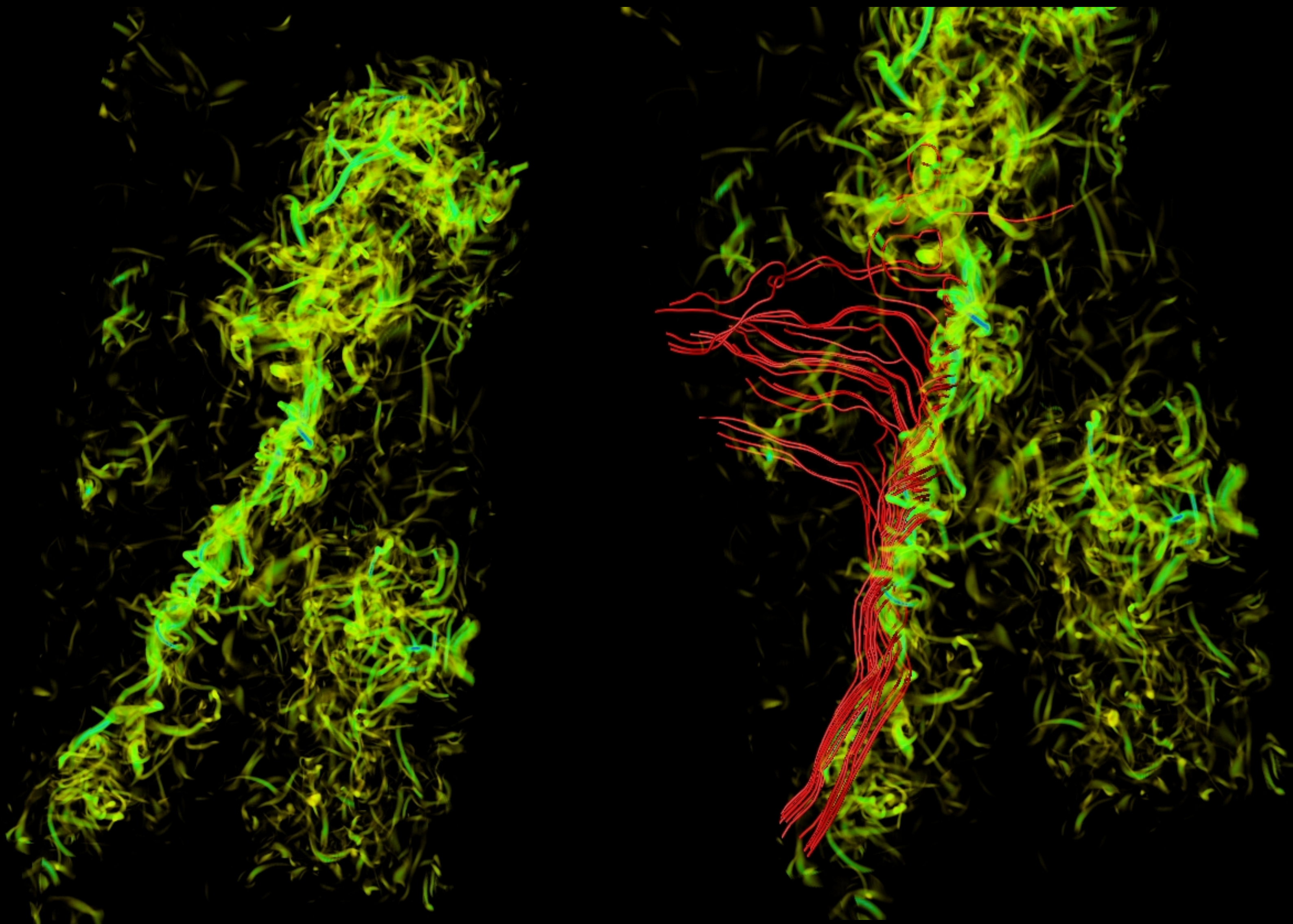
- Fourier representation

$$\mathbf{v}(\mathbf{x}, t) = \int d^3k \; e^{i\mathbf{k}\cdot\mathbf{x}} \; \tilde{\mathbf{v}}(\mathbf{k}, t)$$

- Energy spectrum

$$S(\mathbf{k}) \sim \langle |\mathbf{v}(\mathbf{k})|^2 \rangle$$

- Large, energy containing eddies with correlation scale L. Small scale eddies with wavenumber $k >> 1/L$.



Restitutive forces: gravity and stratification

- Momentum and (potential) temperature equation in the Boussinesq approximation:

$$\begin{aligned}\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} &= -\nabla P - N\theta \boldsymbol{e}_z \\ \partial_t \theta + \mathbf{u}\cdot\nabla\theta - \kappa\Delta\theta &= Nw,\end{aligned}$$

N is the Brunt-Väisälä frequency, w is the vertical component of $\mathbf{u}$.

- Quadratic invariant ($\mathbf{F} = 0$, $\nu = 0$):

$$E = \int \mathbf{u}^2 \, \mathrm{d}^3 x$$

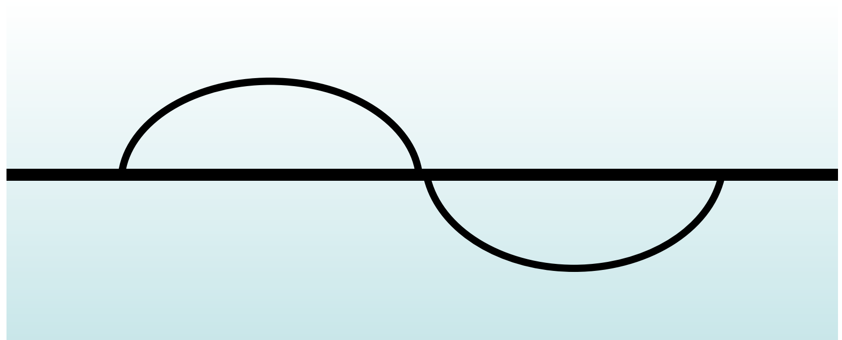
- Froude number:

$$Fr = \frac{u_0}{N L_0}$$

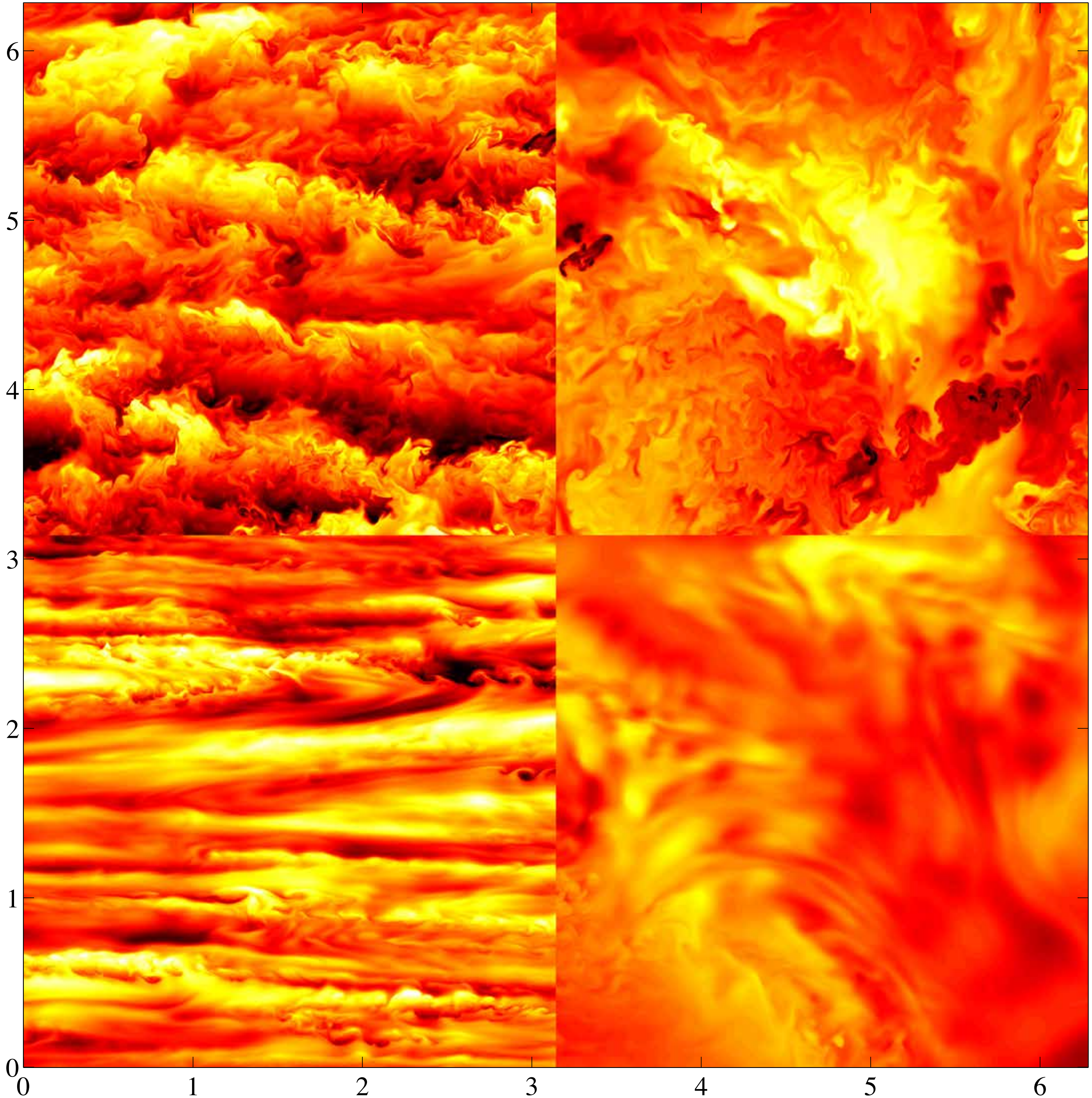
Waves in stratified flows

- Momentum and (potential) temperature equation in the Boussinesq approximation:

$$\partial_t \mathbf{u} + \mathbf{u} \quad \nabla \mathbf{u} - \nu \Delta \mathbf{u} = -\nabla P - N\theta \boldsymbol{e}_z$$
$$\partial_t \theta + \mathbf{u} \quad \nabla \theta - \kappa \Delta \theta = N w$$



$$\omega = \pm N \frac{k_\perp}{k}$$



2048^3
(Rorai, Mininni
& Pouquet 2014)

Restitutive forces: rotation

- Momentum equation

$$\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla \mathcal{P} + \nu \nabla^2 \mathbf{u} + \mathbf{F} \qquad \nabla \cdot \mathbf{u} = 0$$

$\mathbf{\Omega}$ is the angular velocity.

- Quadratic invariants ($\mathbf{F} = 0$, $\nu = 0$):

$$E = \int \mathbf{u}^2 \, \mathrm{d}^3x$$

$$H = \int \mathbf{u} \cdot \boldsymbol{\omega} \, \mathrm{d}^3x \qquad \boldsymbol{\omega} = \nabla \times \mathbf{u}$$

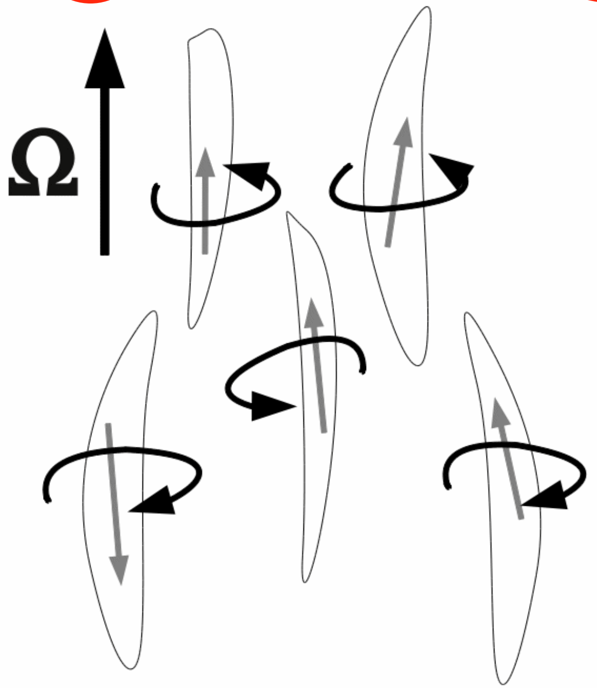
- Reynolds, Rossby, and Ekman numbers

$$Re = \frac{L_F U}{\nu} \qquad Ro = \frac{U}{2\Omega L_F} \qquad Ek = \frac{Ro}{Re} = \frac{\nu}{2\Omega L_F^2}$$

where L_F is the forcing scale.

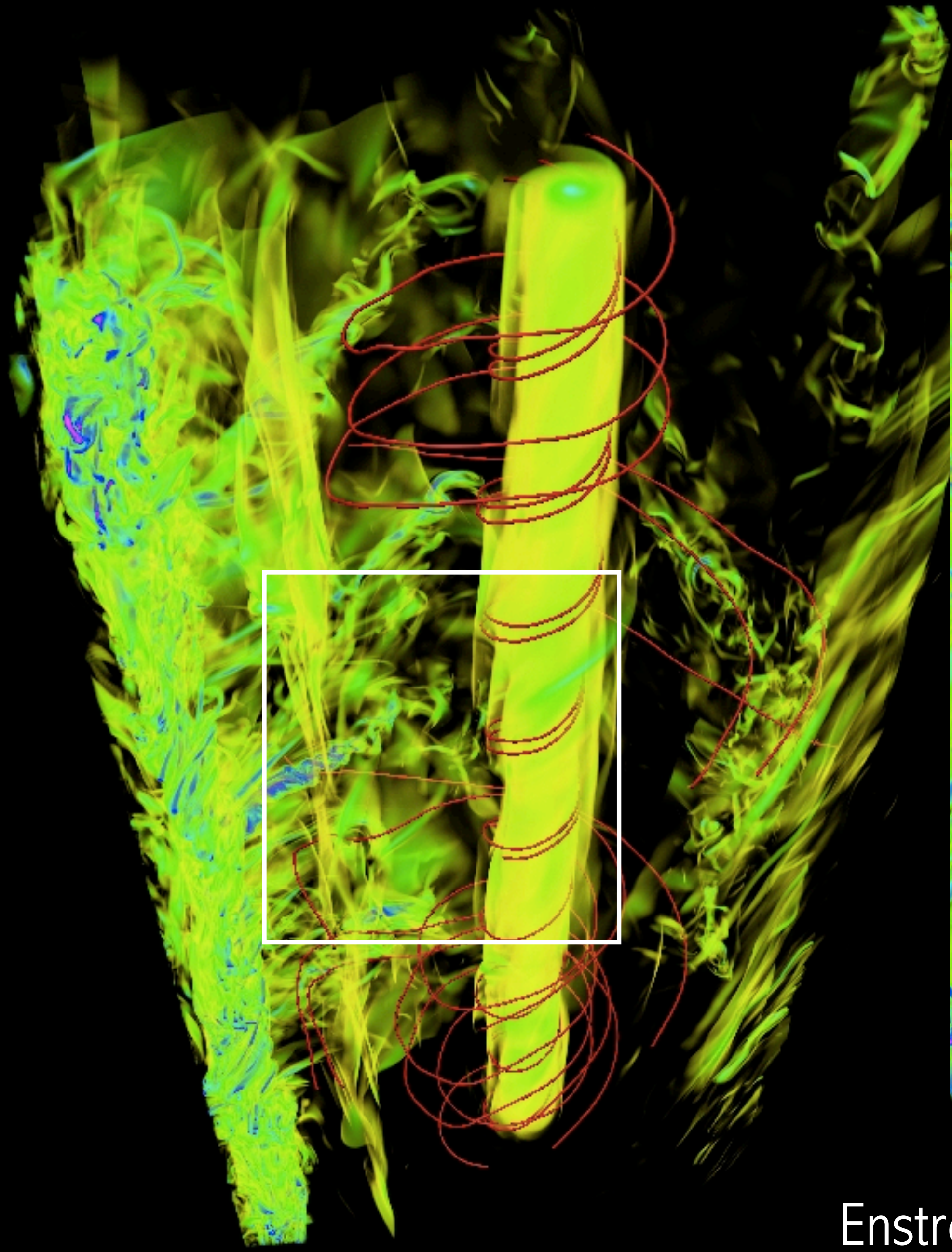
Waves in rotating flows

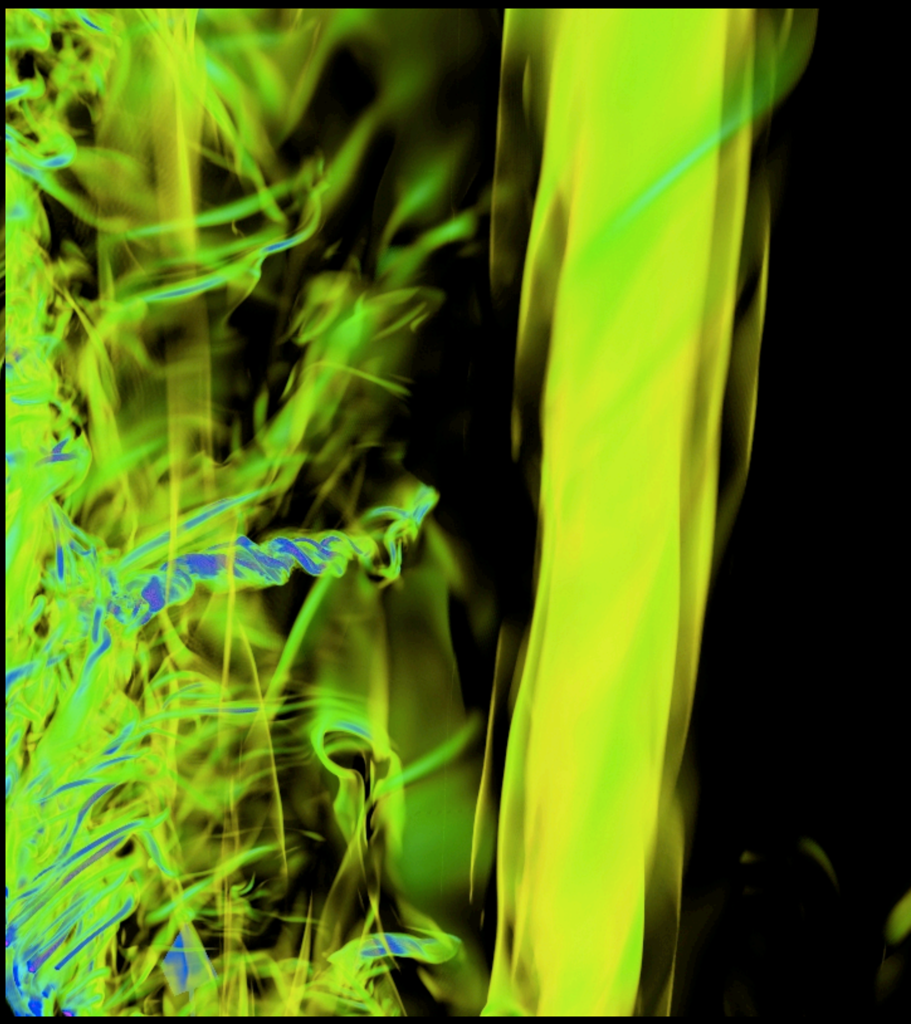
$$\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\omega} \times \mathbf{u} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla \mathcal{P} + \nu \nabla^2 \mathbf{u} + \mathbf{F}$$



$$\omega = \pm \Omega \frac{k_z}{k} \qquad u_x = \pm i u_y$$

Mininni & Pouquet, PRE **79**, 026304 (2009), Phys. Fluids **22**, 035105 (2010), JFM **699**, 263 (2012)





Enstrophy

1536^3

Energy transfer and triadic interactions

- We can decompose the velocity field as

$$\mathbf{u}(\mathbf{k},t) = a_+(\mathbf{k},t)\mathbf{h}_+ + a_-(\mathbf{k},t)\mathbf{h}_-$$

$$a_s(\mathbf{k},t) = A_s(T)e^{i\omega_{\mathbf{k}}t}$$

Anisotropy and time scales

Time scales:

- Wave period

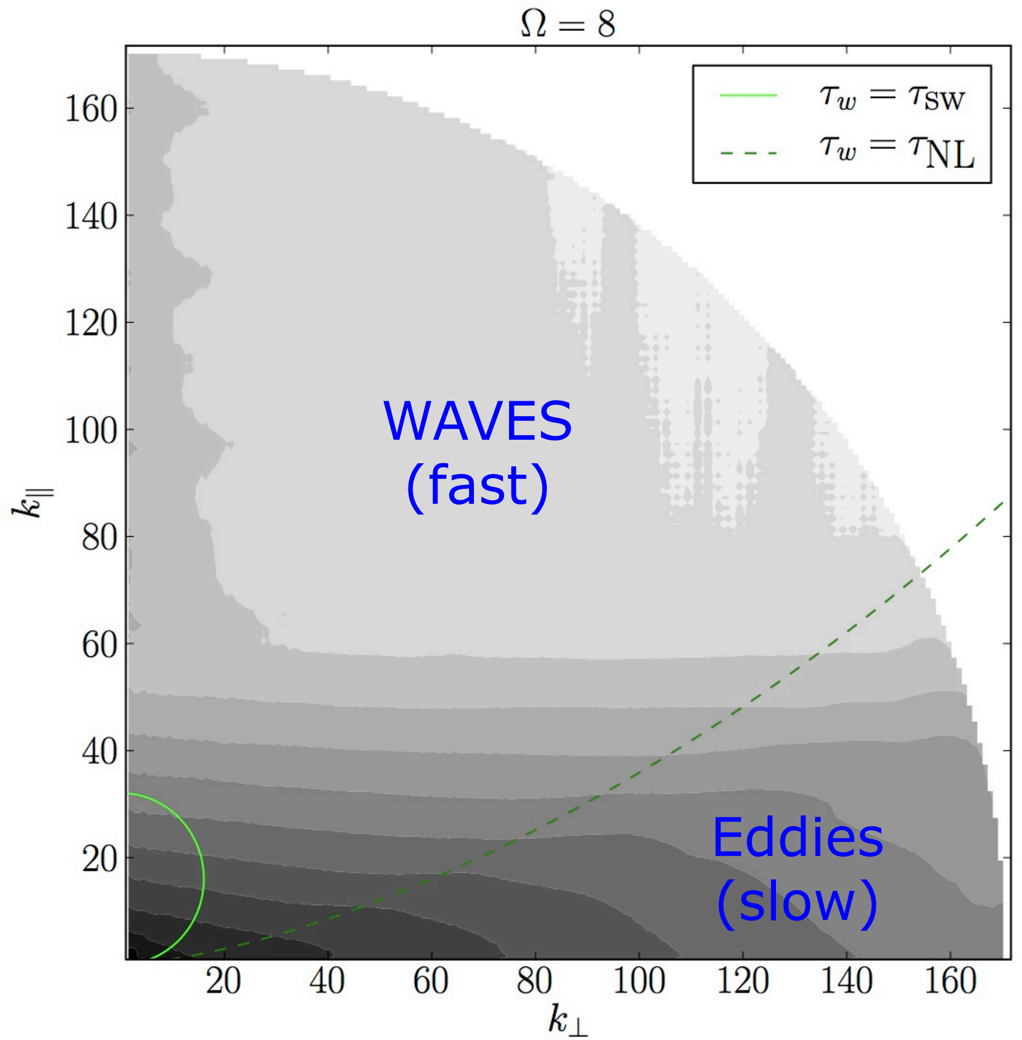
$$\tau_\omega(\mathbf{k}) = C_\omega \frac{k}{2\Omega k_{||}}$$

- Non-linear time

$$\tau_{\rm NL}(\mathbf{k}) = C_{\rm NL} \frac{1}{\epsilon^{1/4}\Omega^{1/4}k^{1/2}}$$

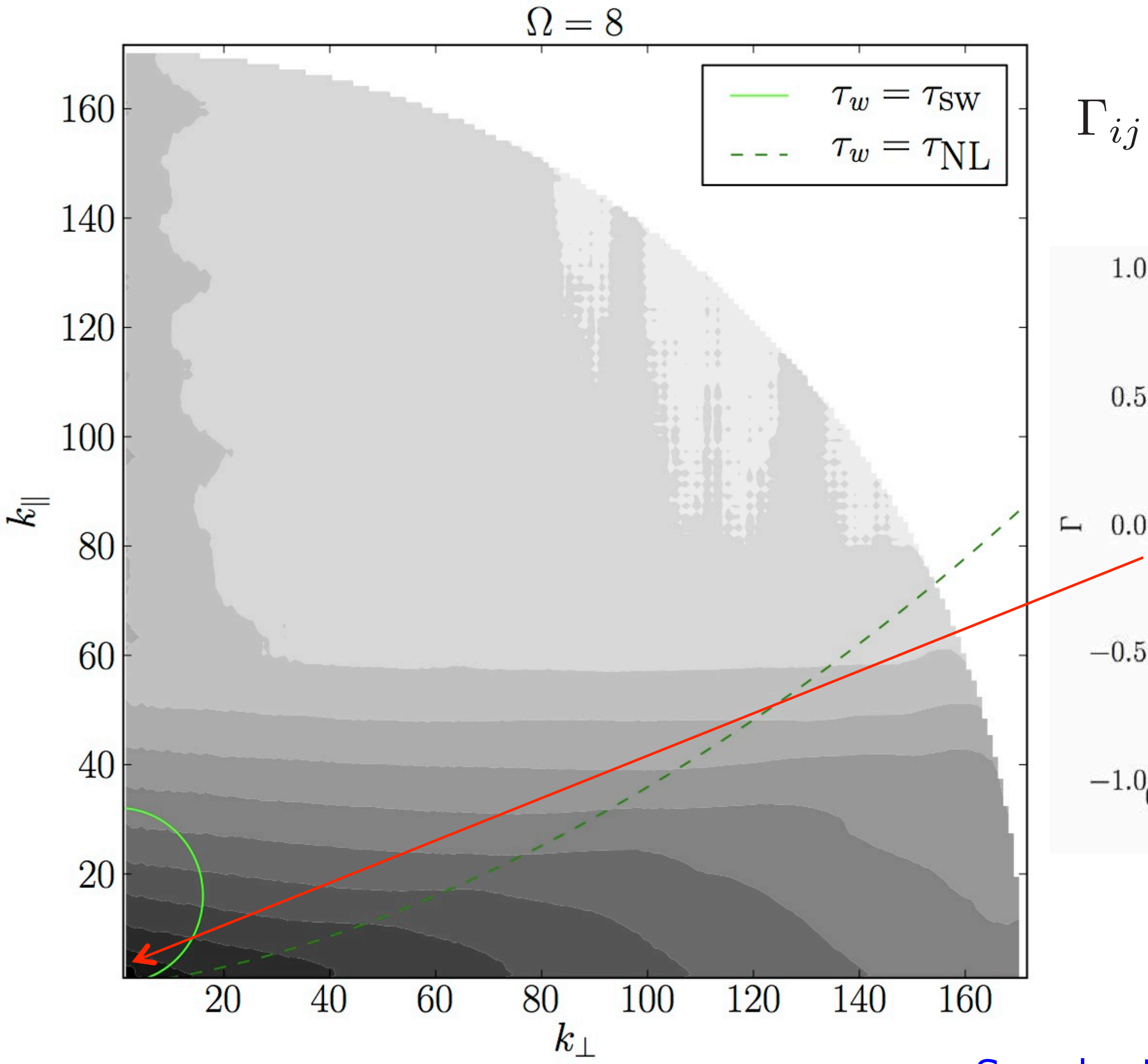
- Sweeping time

$$\tau_{\rm sw}(\mathbf{k}) = C_{\rm sw} \frac{1}{Uk}$$

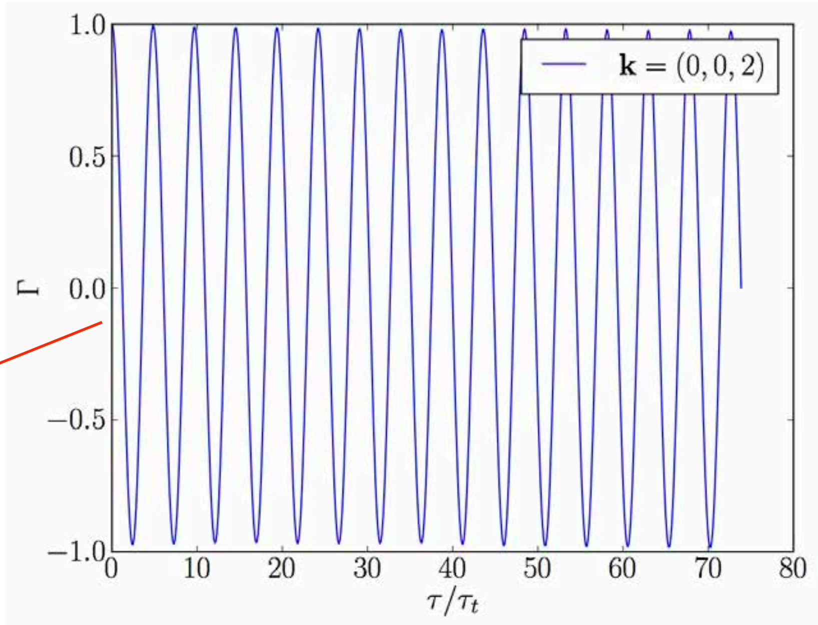


Clark di Leoni, Cobelli, Mininni, Dmitruk & Matthaeus (2014)

Correlation times

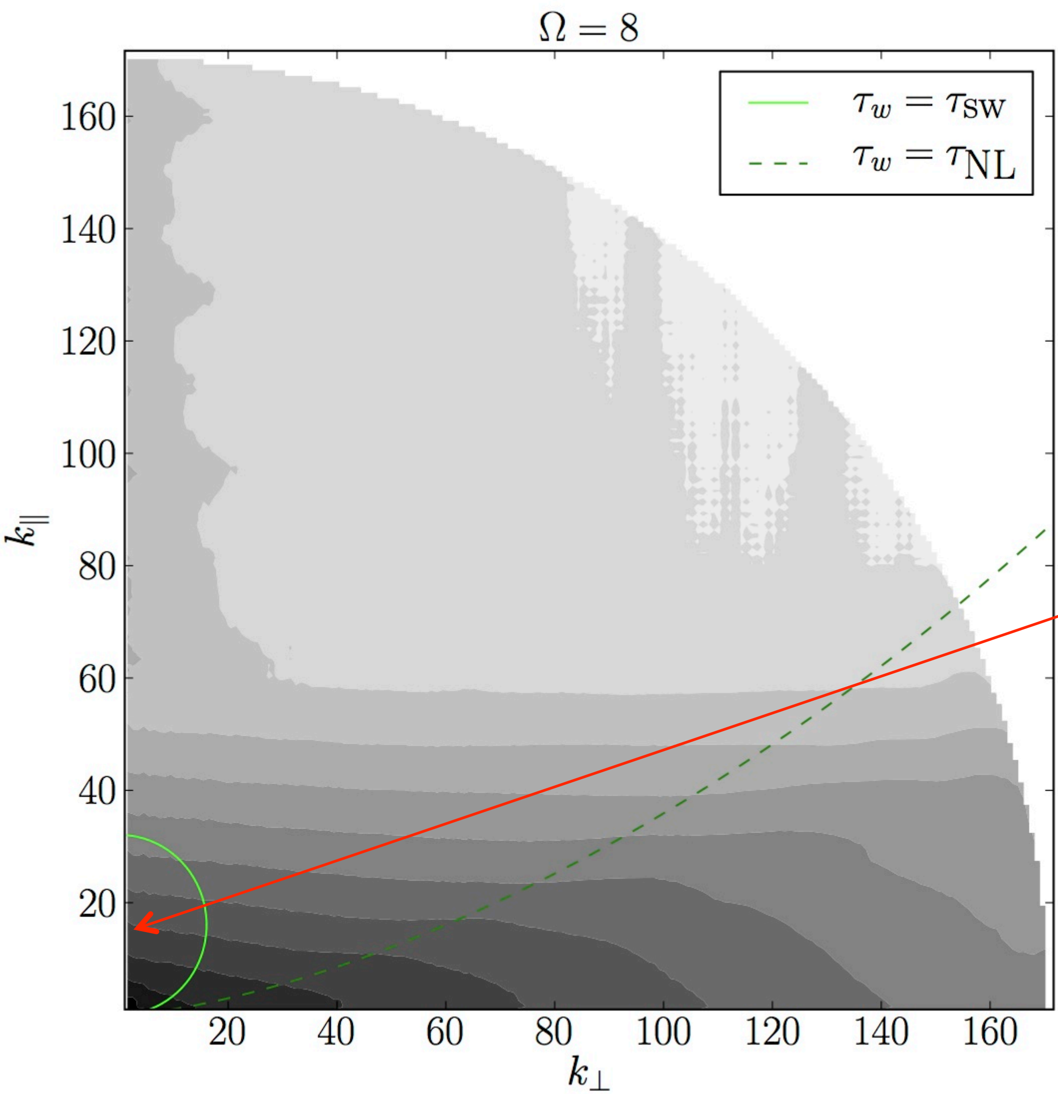


$$\Gamma_{ij}(\mathbf{k}, \tau) = \frac{\langle \hat{u}_i^*(\mathbf{k}, t)\hat{u}_j(\mathbf{k}, t+\tau)\rangle_t}{\langle |\hat{u}_i^*(\mathbf{k}, t)\hat{u}_j(\mathbf{k}, t)|\rangle_t}$$

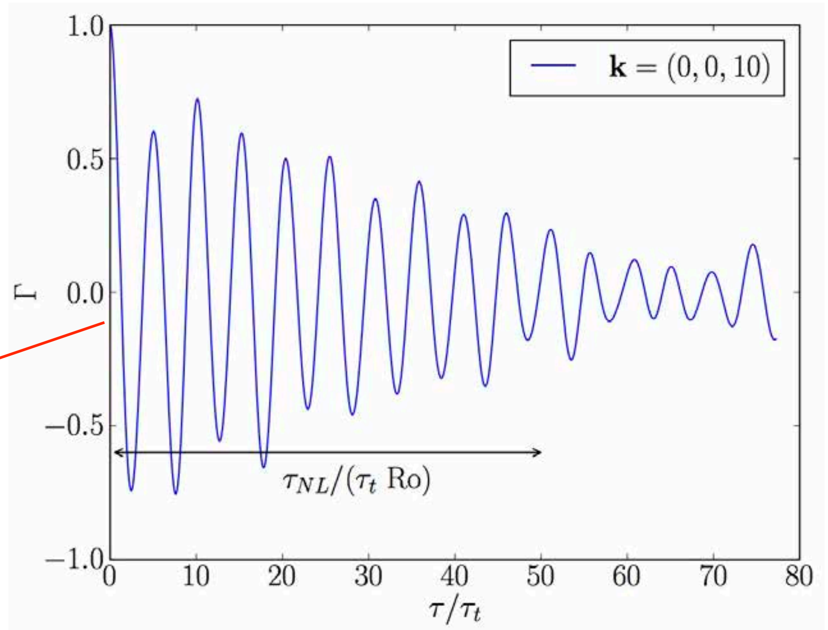


See also Fabier, Godeferd and Cambon (2010)

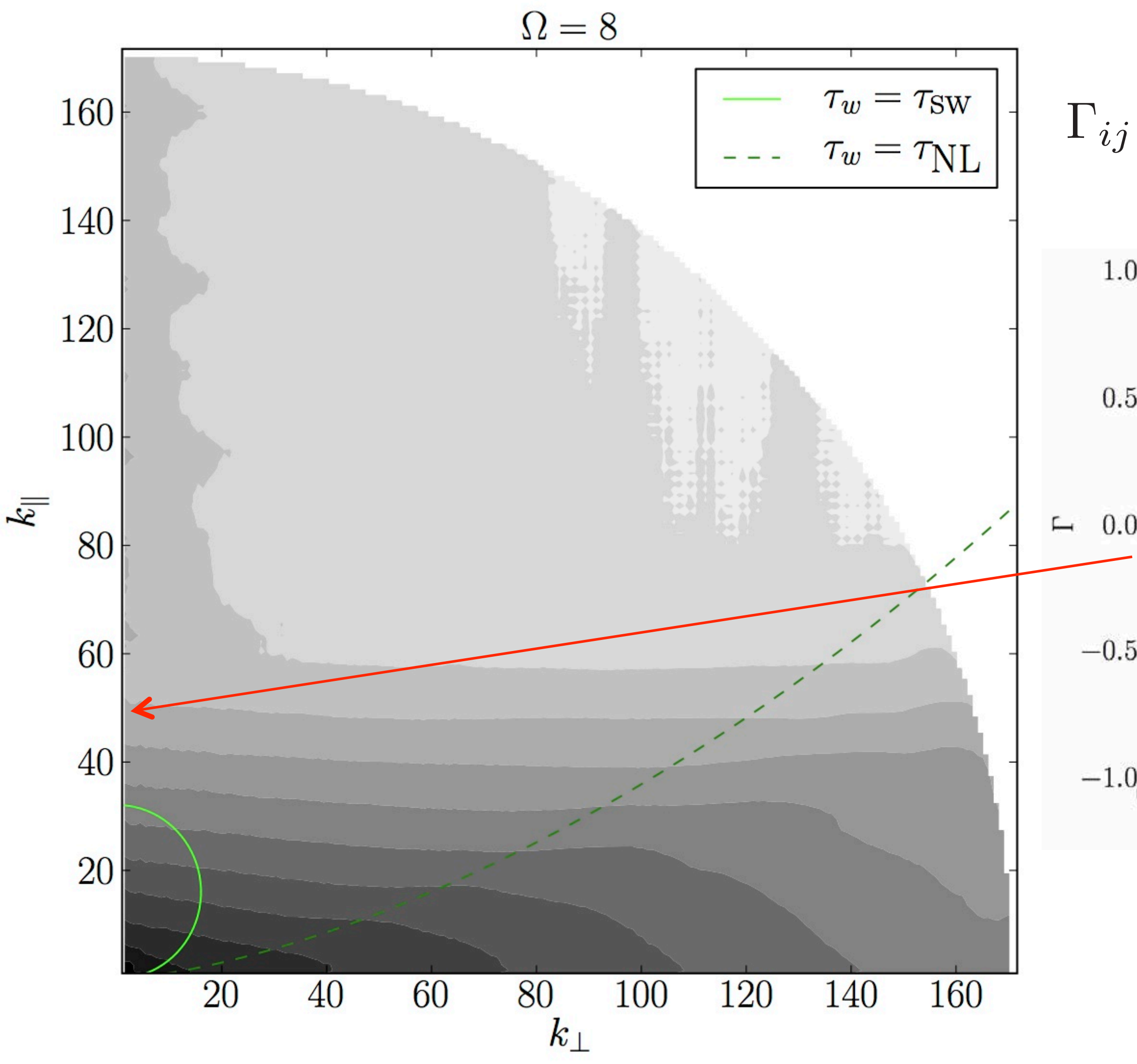
Correlation times



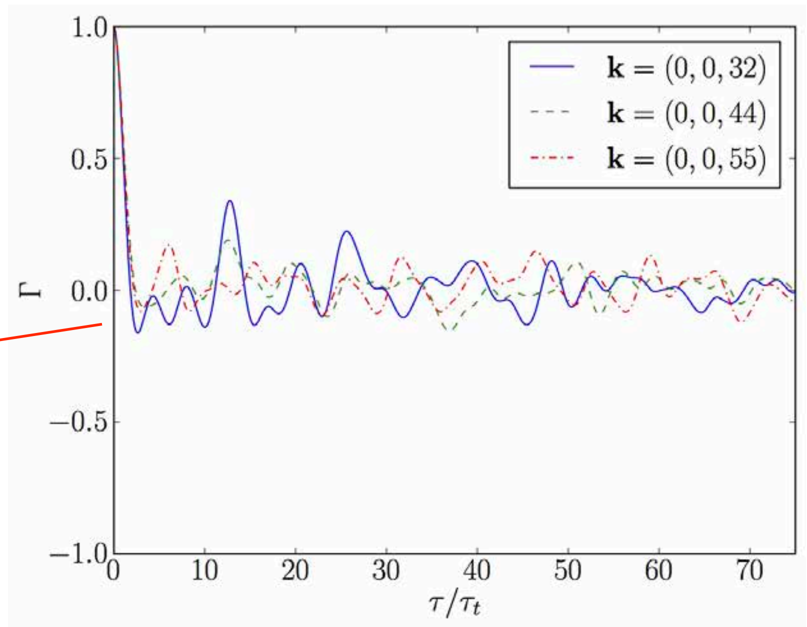
$$\Gamma_{ij}(\mathbf{k},\tau) = \frac{\langle \hat{u}_i^*(\mathbf{k},t)\hat{u}_j(\mathbf{k},t+\tau)\rangle_t}{\langle |\hat{u}_i^*(\mathbf{k},t)\hat{u}_j(\mathbf{k},t)|\rangle_t}$$



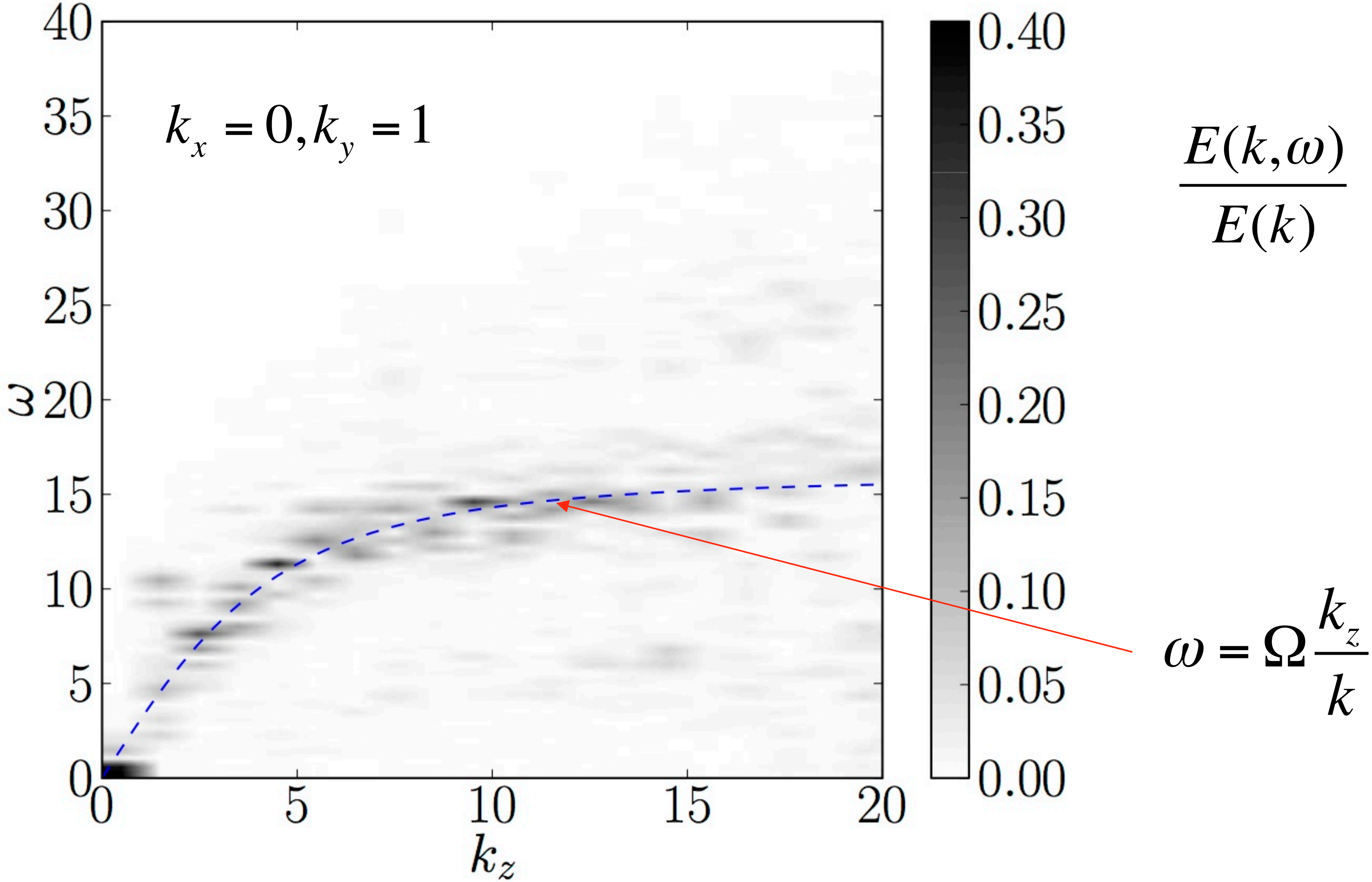
Correlation times



$$\Gamma_{ij}(\mathbf{k},\tau) = \frac{\langle \hat{u}_i^*(\mathbf{k},t)\hat{u}_j(\mathbf{k},t+\tau)\rangle_t}{\langle|\hat{u}_i^*(\mathbf{k},t)\hat{u}_j(\mathbf{k},t)|\rangle_t}$$



Waves or eddies?



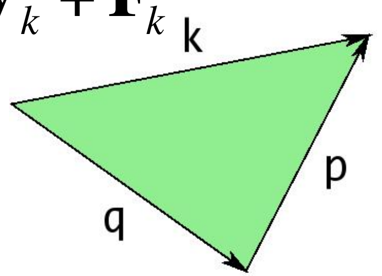
Clark di Leoni, Cobelli, Mininni, Dmitruk & Matthaeus (2014)
See also Hopfinger et al 1982; Bewley et al 2007; Bordes, Moisy, Dauxois, and Cortet 2012

Triadic interactions in rotating turbulence

- The evolution of each velocity mode in Fourier space is

$$\frac{\partial \mathbf{v}_k}{\partial t} = -\int_{p,q} \left[\left(\mathbf{v}_p \cdot \nabla \right) \mathbf{v}_q \right] dpdq - i\mathbf{k}P_k - \nu k^2 \mathbf{v}_k + \mathbf{F}_k$$

$$\mathbf{k} + \mathbf{p} + \mathbf{q} = 0$$



- In rotating flows we have Rossby waves, that slow down the energy transfer through resonant interactions (Cambon and Jacquin 1989, Cambon, Mansour, and Godeferd 1997).

$$u_k \rightarrow A_{s,k} e^{i\omega_{s,k} t}$$

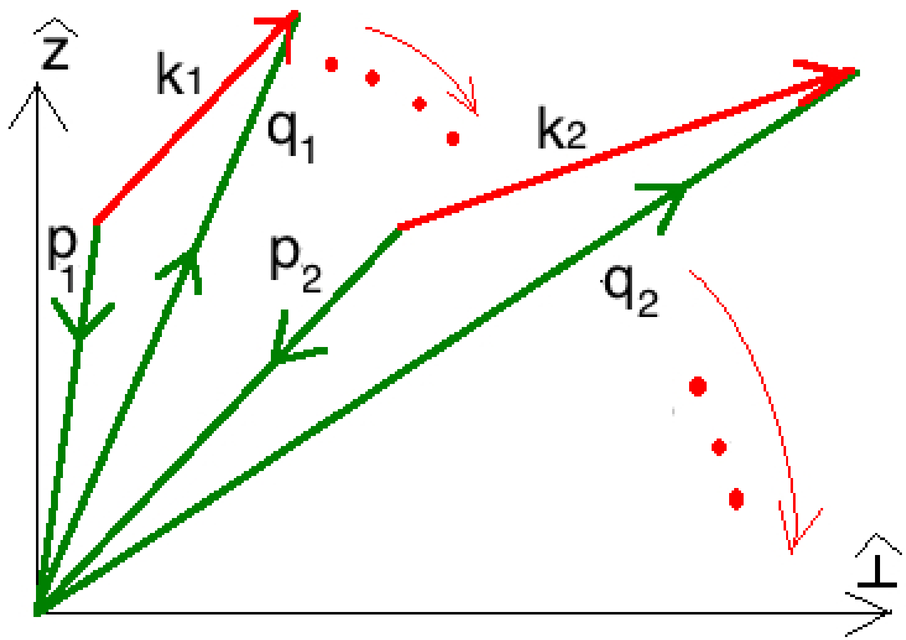
$$\int_{p,q} \left[\left(\mathbf{v}_p \cdot \nabla \right) \mathbf{v}_q \right] dpdq \rightarrow \int_{p,q} \left[\left(A_{s,p} \cdot \nabla \right) A_{s,q} \right] e^{i(\omega_{s,k}+\omega_{s,p}+\omega_{s,q})t}$$

$$\omega_{s,k} + \omega_{s,p} + \omega_{s,q} = s_k \frac{k_z}{k} + s_p \frac{p_z}{p} + s_q \frac{q_z}{q} = 0$$

Energy transfer and triadic interactions

$$\partial_t a^{s_k}(t) = \mathcal{R}o \sum_{s_p, s_q} \int_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} C_{kpq}^{s_k s_p s_q} a^{s_p^\star} a^{s_q^\star} \, e^{i(\omega_{s_k}+\omega_{s_p}+\omega_{s_q})t} dp dq$$

$$s_k \frac{k_{||}}{k} + s_p \frac{p_{||}}{p} + s_q \frac{q_{||}}{q} = \mathcal{O}(\mathcal{R}o)$$



- Instability theorem (Waleffe 1993).
- However, this is not valid for too small values of k_z.
- See Lamriben, Cortet & Moisy 2011 for an experimental study of anisotropic transfer.

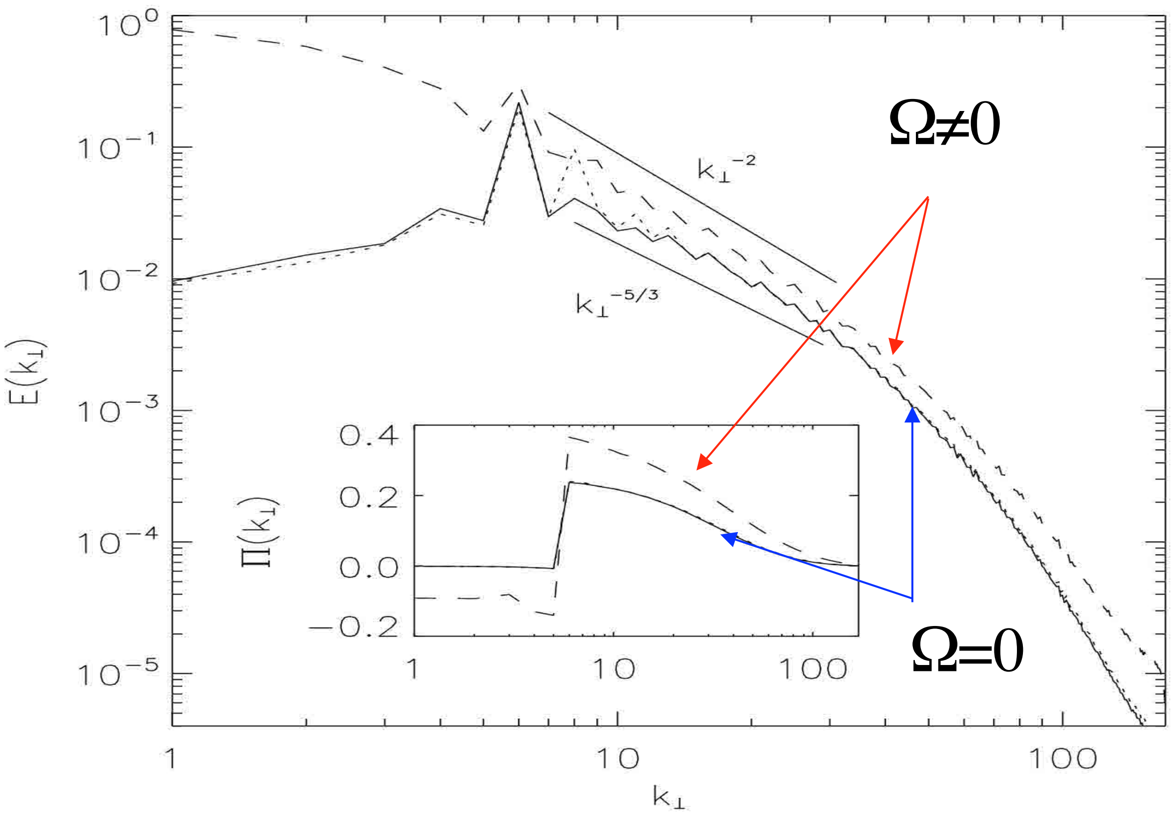
Phenomenology of rotating turbulence

- The interaction of waves and eddies slows down the cascade (Cambon and Jacquin 1989; Cambon, Mansour, and Godeferd 1997).
- Following Kraichnan (1965) phenomenology, we can assume that the time to move energy across scales is increased by a factor τ_l / τ_Ω.
- The inverse of the transfer time then becomes $1/\tau_{NL} = \tau_\Omega / \tau_l^2$.
- As a result of the resonant interactions, the flow also becomes anisotropic, with $1/\tau_l \sim u_l / l_\perp$.
- The energy transferred between scales per unit of time is

$$\varepsilon \sim u_l^2/\tau_{NL} \sim u_l^4/l_\perp^2, \text{ and } u_l^2 \sim l_\perp.$$

- Then the energy spectrum is $E(k_\perp) \sim k_\perp^{-2}$ (Dubrulle 1992, Zhou 1995)
- A more elegant derivation can be found in Cambon and Jacquin (1989).

Energy spectrum in rotating flows



Non-helical case:

- An inverse cascade of energy develops for small *Ro*.
- The flow becomes anisotropic.
- The spectrum goes towards $k_\perp^{-2}$.

Mininni, Alexakis & Pouquet, PoF **21**, 015108; Mininni & Pouquet, PRE **79**, 026304 (2009)

Helicity as an invariant of 3D Euler

- Inertial waves are helical! What happens when they are not balanced?
- Euler equations for an ideal, incompressible fluid with uniform density (1757):

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p$$

- The equations can be written as

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \omega\times\mathbf{u}\right) = -\nabla p'$$

with $\boldsymbol{\omega} = \nabla\times\mathbf{u}$

- Note that when $\boldsymbol{\omega}\times\mathbf{u} = 0$

the non-linear term becomes zero.

Helical flows

$$H = \int \boldsymbol{\omega} \cdot \mathbf{u} \, dV \qquad \boldsymbol{\omega} = \nabla \times \mathbf{u}$$

- When maximal, $\boldsymbol{\omega} \times \mathbf{u} = 0$
- Helicity is thus associated with corkscrew motions.
- As the non-linear term in the momentum equation becomes zero or negligible, helical flows are extremelly stable.

Helicity was discovered "recently"

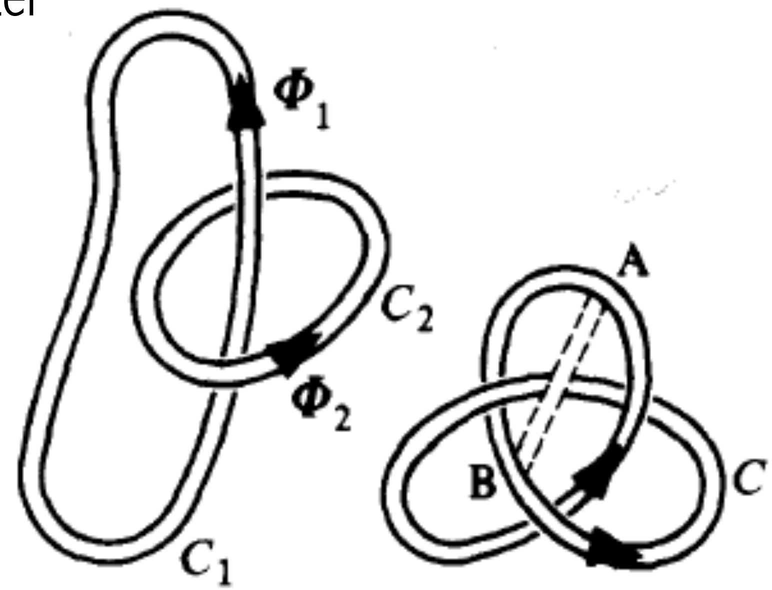
- In 1958 Woltjer introduces the magnetic helicity (later studied by Chandrasekhar and Kendall):

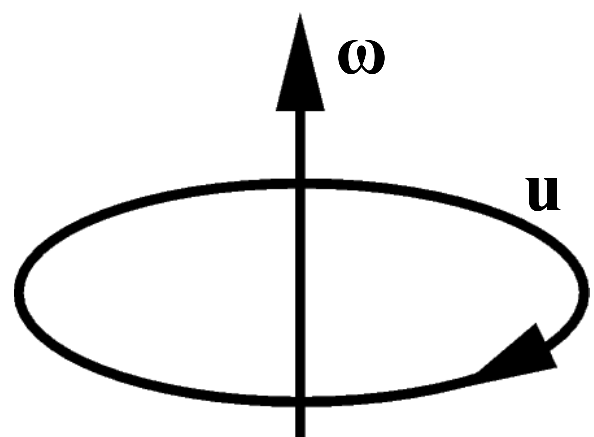
$$H_m = \int \mathbf{B} \cdot \mathbf{A} dV \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

- In 1967, Moffatt finds its hydrodynamic equivalent:

$$H = \int \boldsymbol{\omega} \cdot \mathbf{u} dV \qquad \boldsymbol{\omega} = \nabla \times \mathbf{u}$$

- Helicity is zero for 2D flows, and it is a conserved quantity in 3D hydrodynamics (without and with rotation).
- Helicity measures the structural complexity of the flow: it is proportional to the number of *links* in the field lines.
- What is the role of helicity in atmospheric, geophysical, and astrophysical flows?

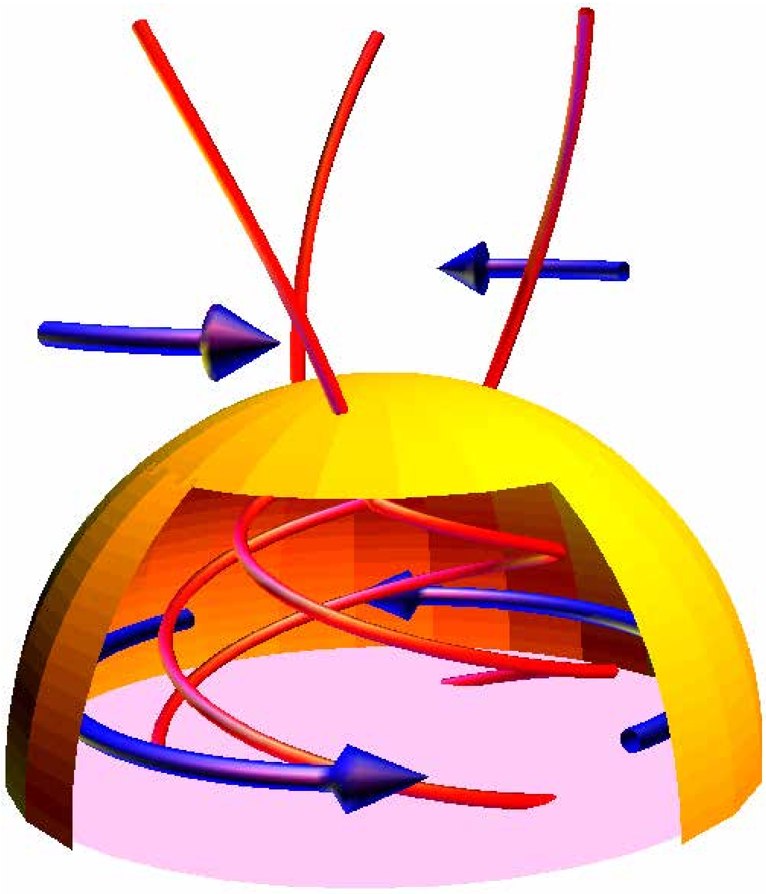




The role of helicity

Helical flows are relevant for many applications:

- Solar and geophysical dynamo: helical flows are known to sustain large-scale dynamo action (Parker 1955, Pouquet et al. 1976, Krause & Rädler 1986).
- Helical velocity fields result in the "alpha-effect", and in the generation of magnetic fields by self-induction.
- The large-scale magnetic fields generated by this mechanism are helical (Titov & Demoulin).
- The mechanism is also relevant in the presence of kinetic effects (Mininni, Gómez & Mahajan 2003)

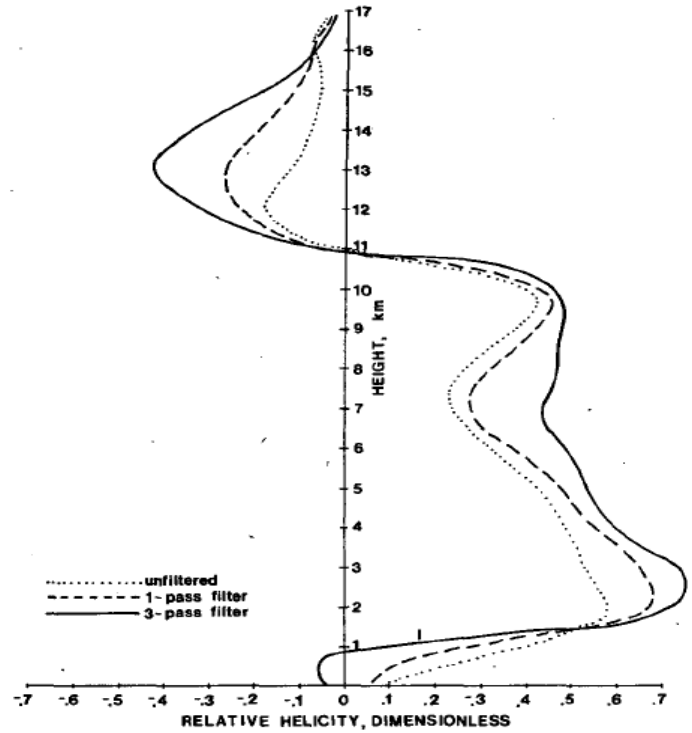


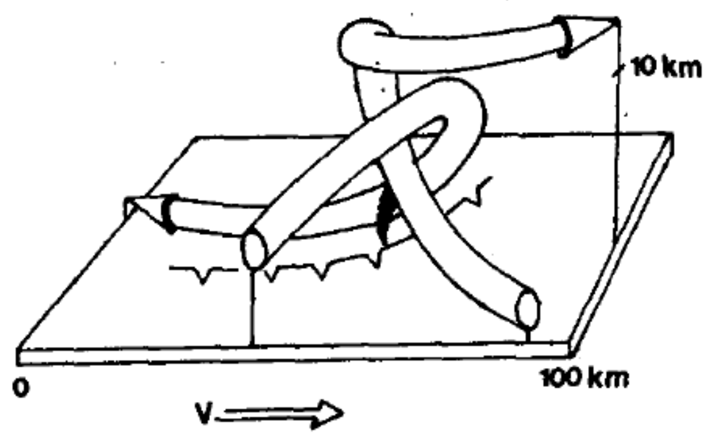
Berger (1999)

The role of helicity

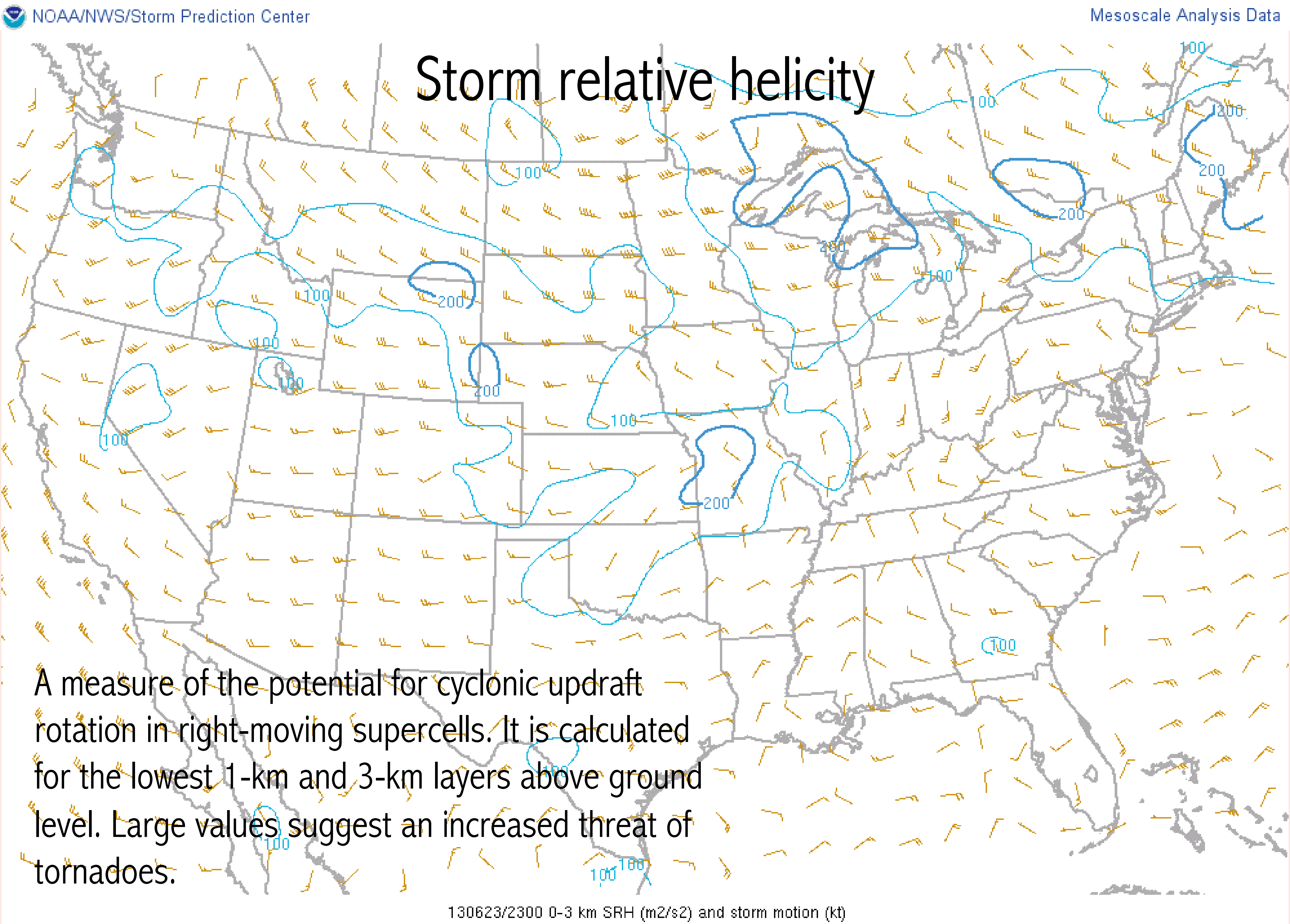
Helical flows are relevant for many applications:

- Atmospheric flows: Lilly (1986) speculated that rotating convective supercell storms are more stable because flows are helical.
- Some authors claim that helicity may play a role in the self-organization of the flow leading to formation of tornadoes (Montgomery 2006, Levina 2013).
- Indices based on helicity are used for forecasting purposes.



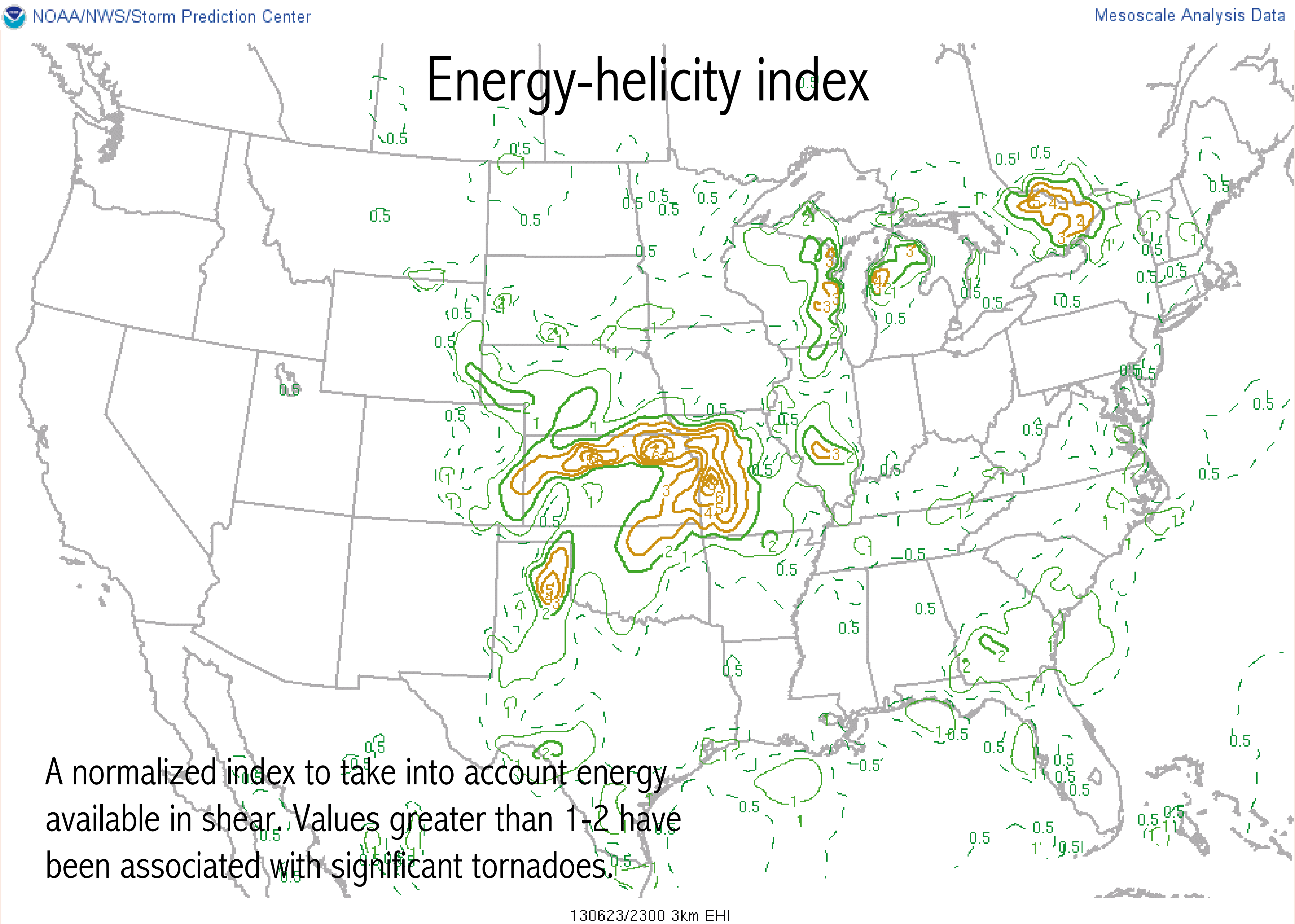


Storm relative helicity



A measure of the potential for cyclonic updraft rotation in right-moving supercells. It is calculated for the lowest 1-km and 3-km layers above ground level. Large values suggest an increased threat of tornadoes.

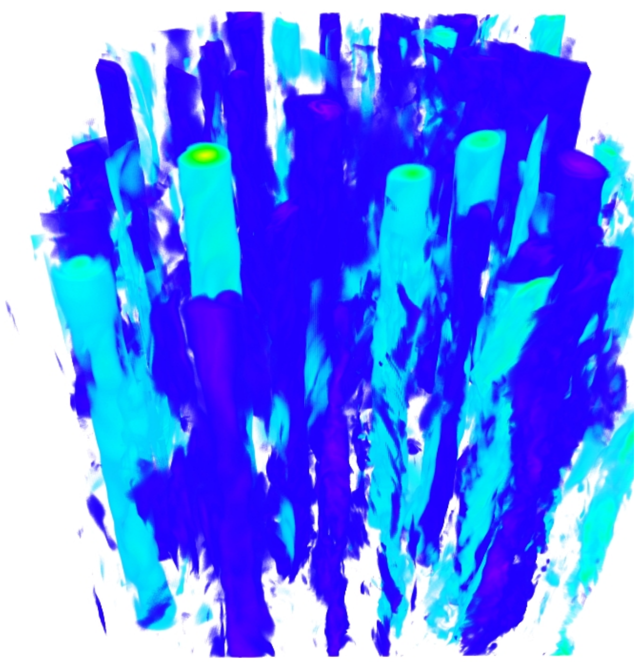
Energy-helicity index

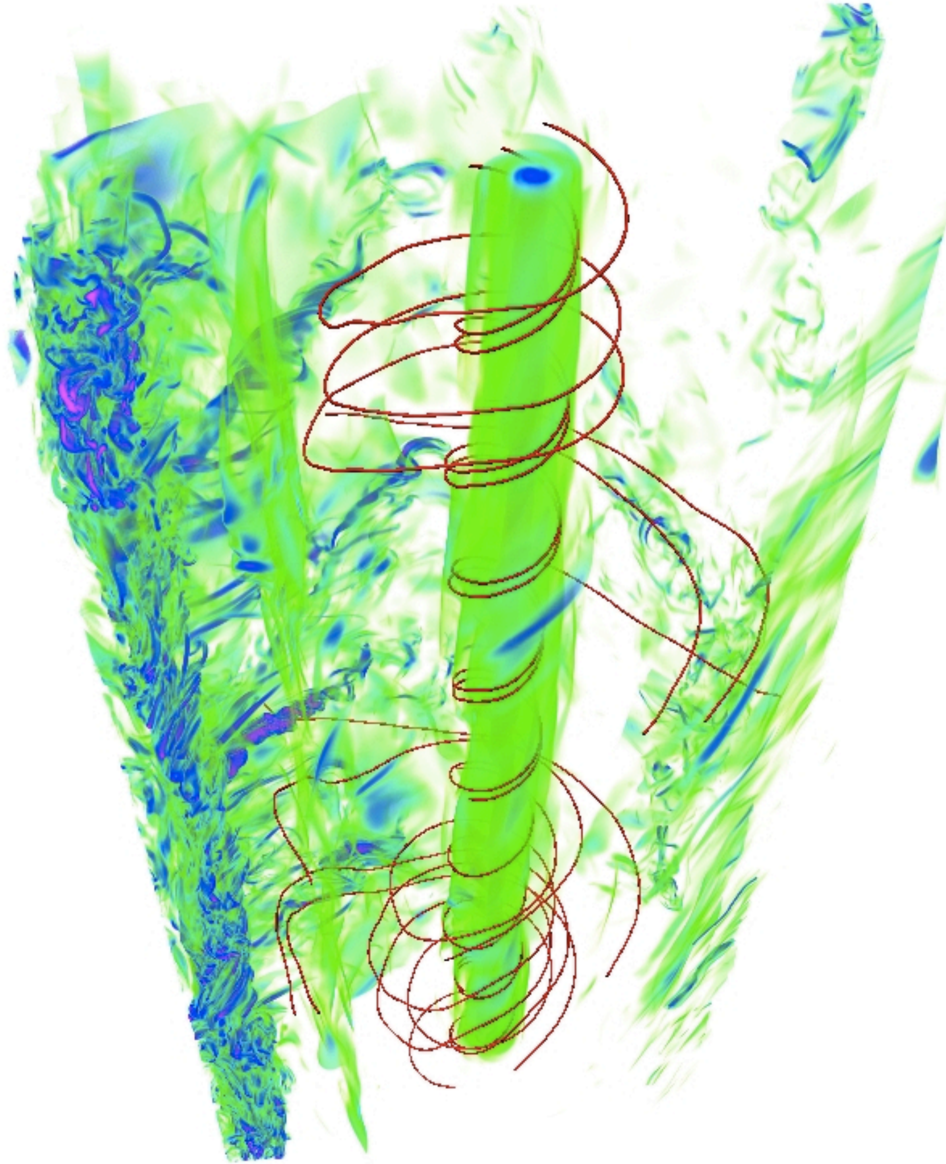


A normalized index to take into account energy available in shear. Values greater than 1-2 have been associated with significant tornadoes.

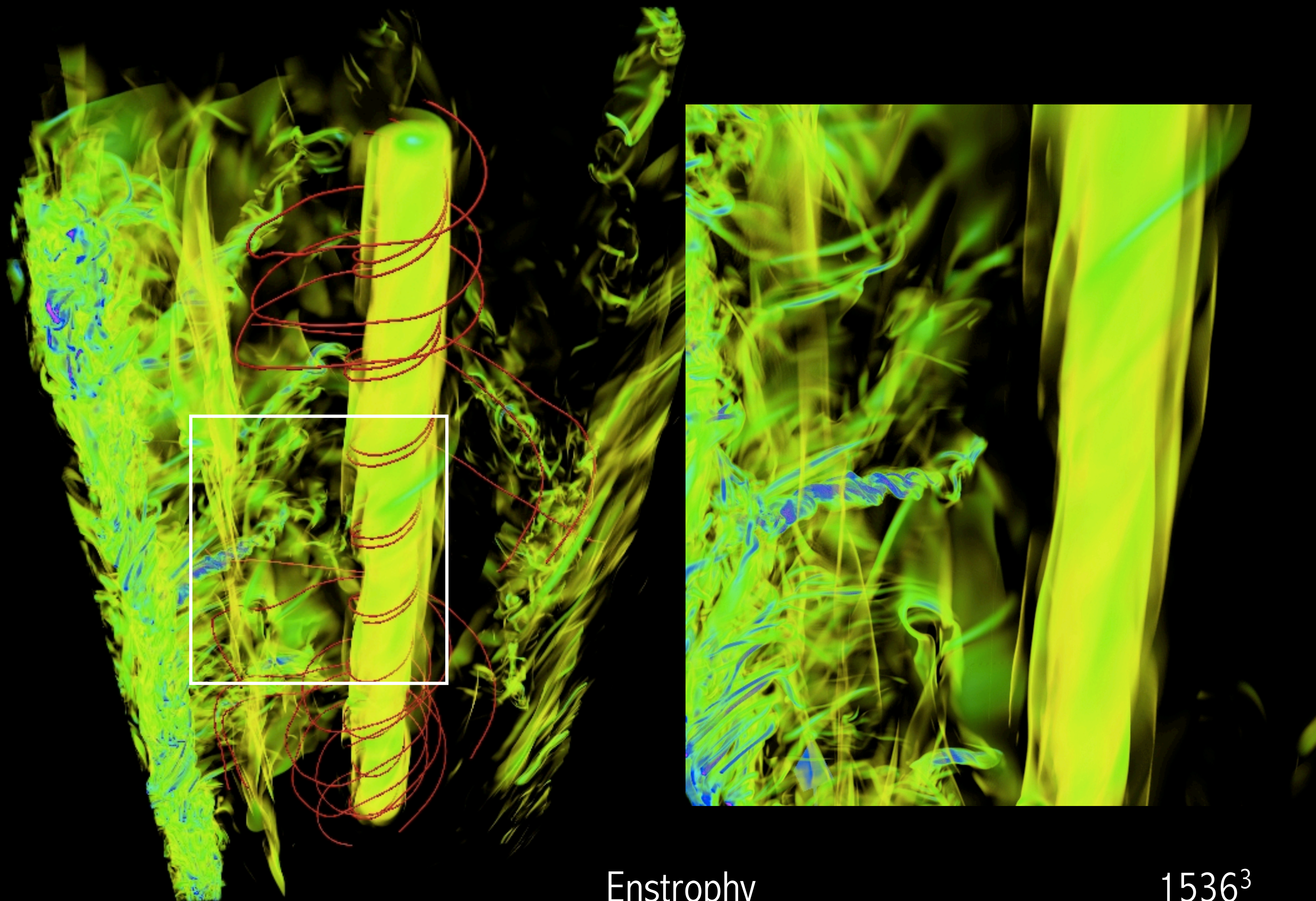
Helical rotating turbulence

- 512^3 to 3072^3 spatial resolutions.
- *Re* up to 10000, *Ro* down to 0.06.
- Laminar column-like structures develop in the flow.
- Structures are helical and stable.

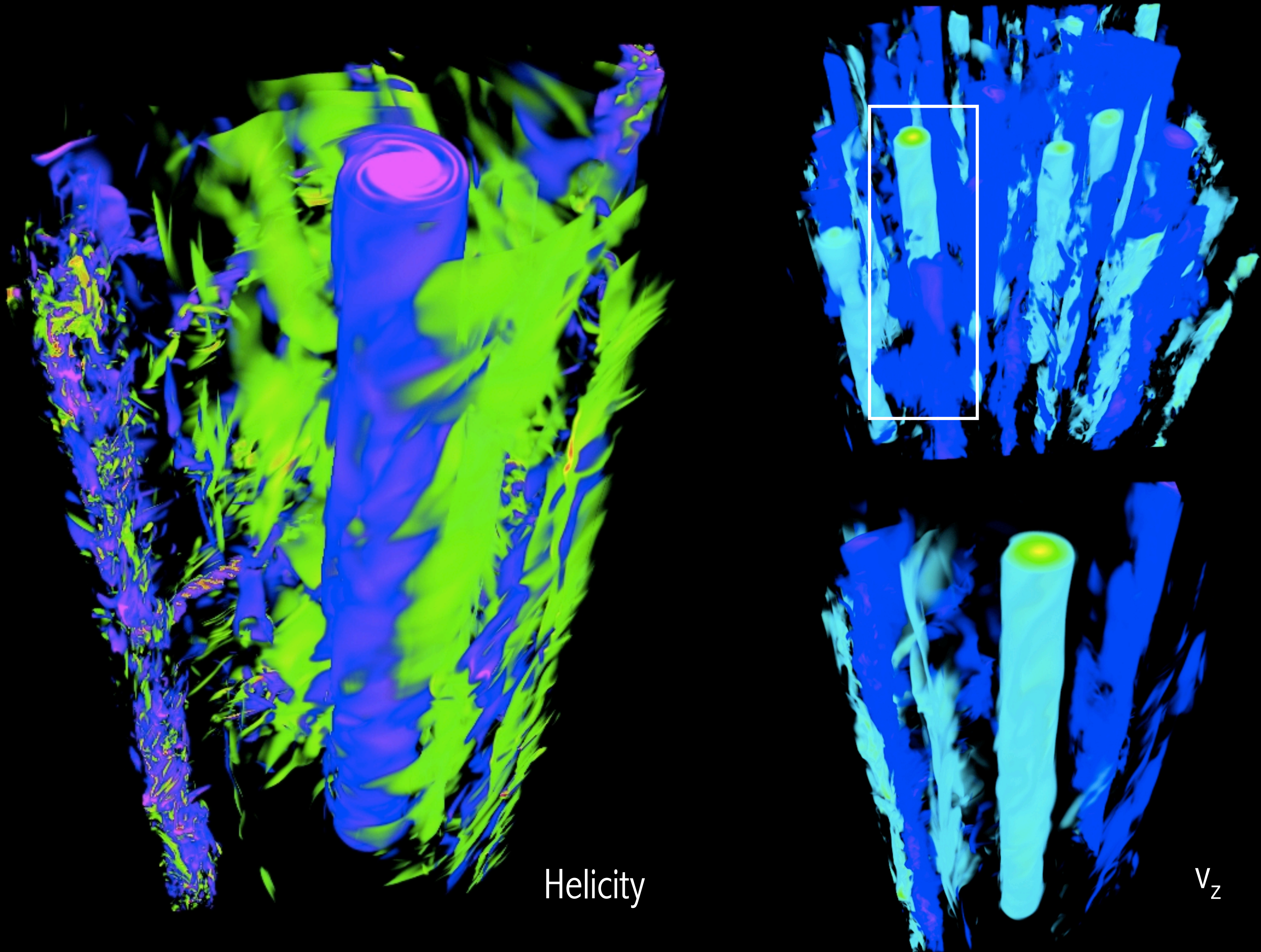


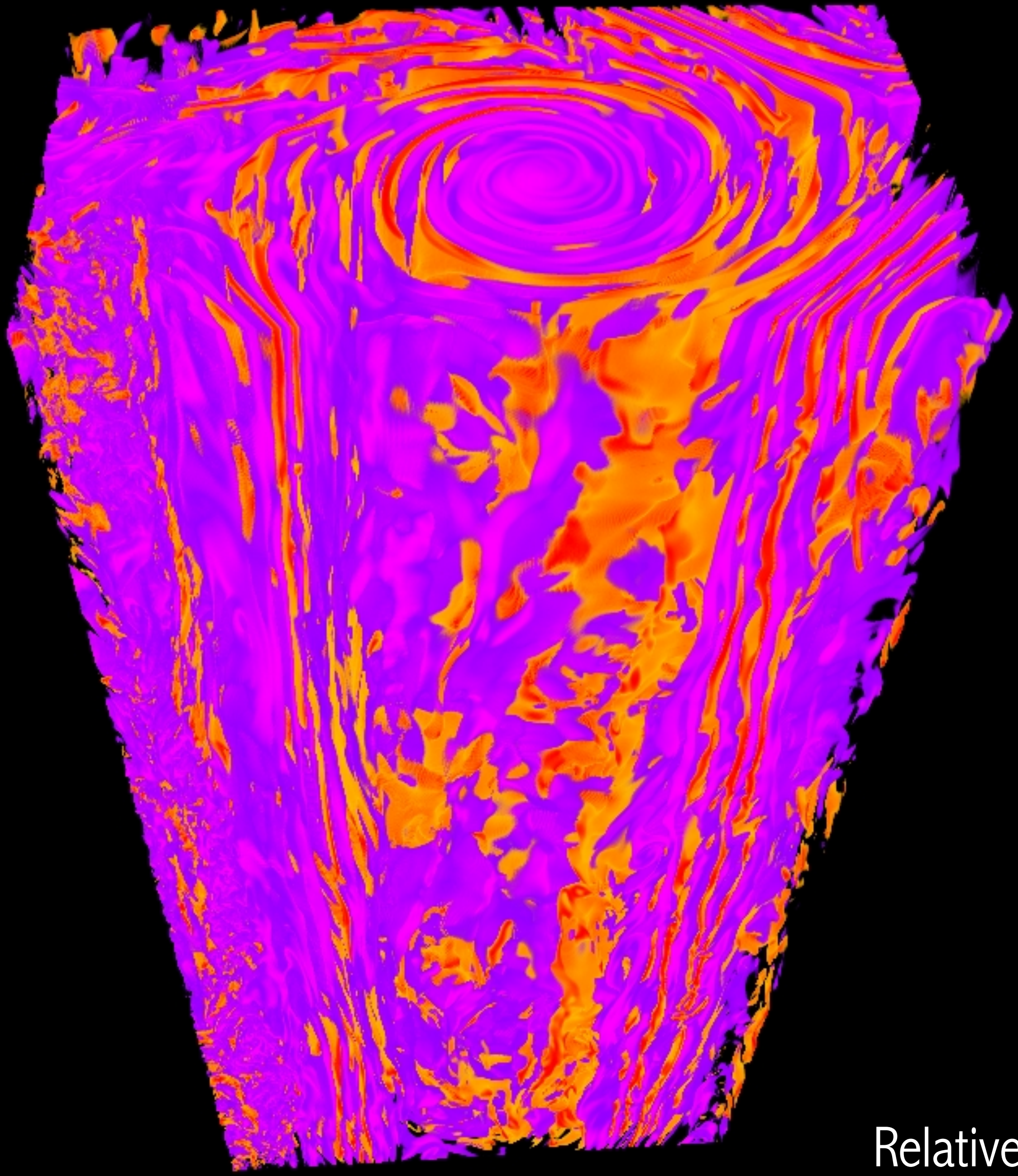


Mininni & Pouquet, PRE **79**, 026304 (2009), Phys. Fluids **22**, 035105 (2010), JFM **699**, 263 (2012)

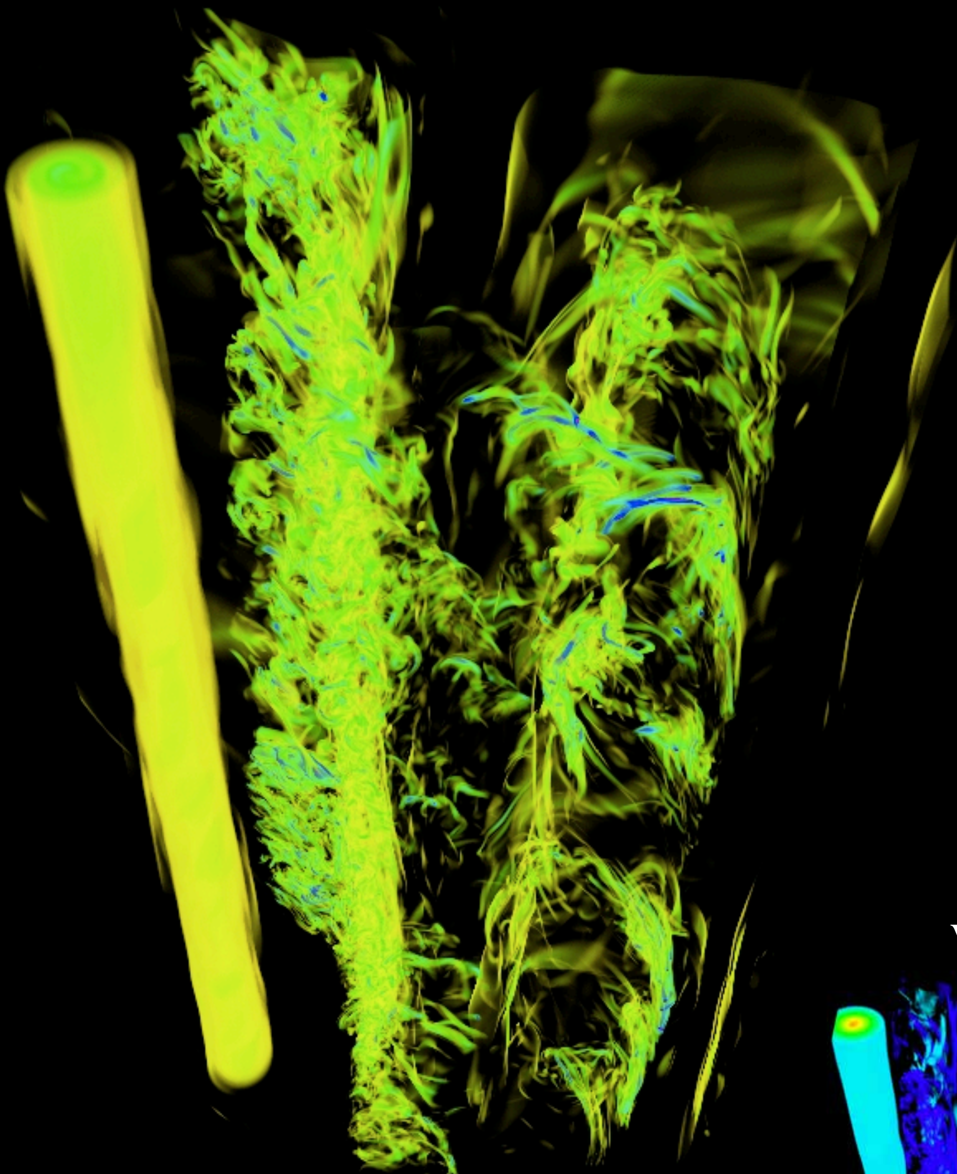


Enstrophy 1536^3


Helicity
v_z

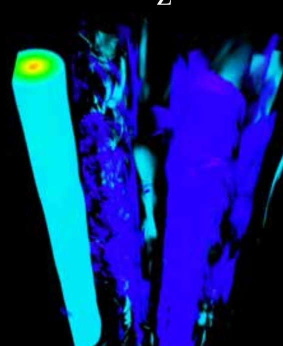


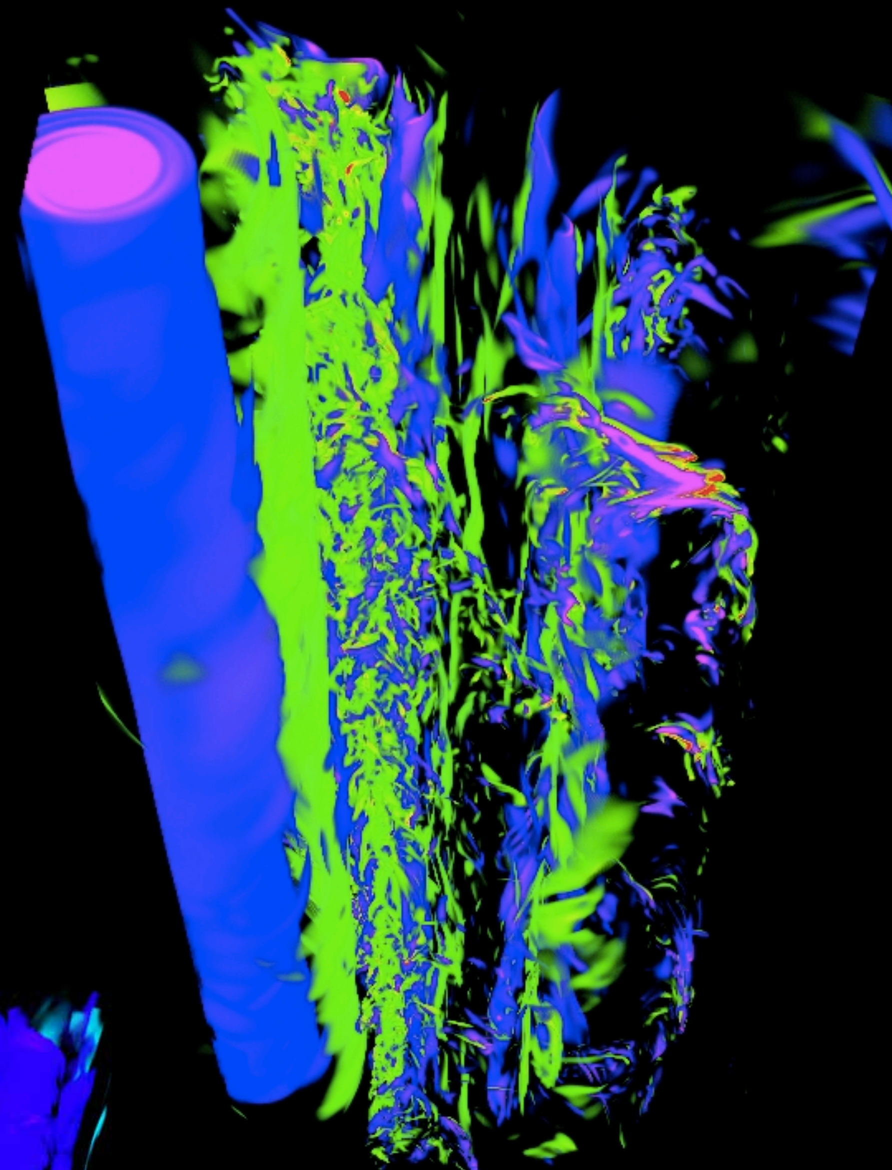
Relative helicity



Helicity

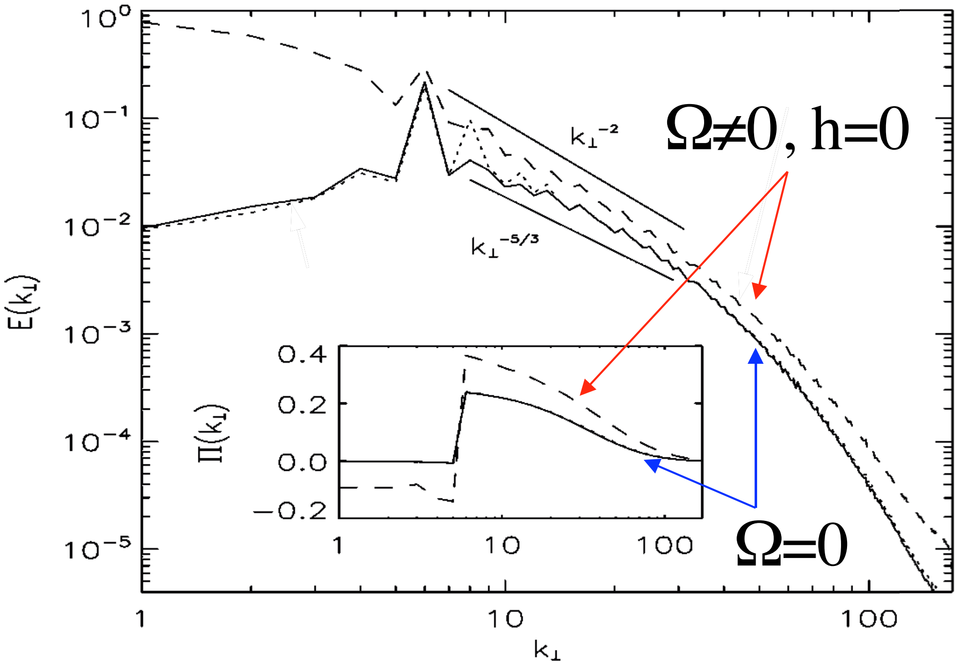
v_z

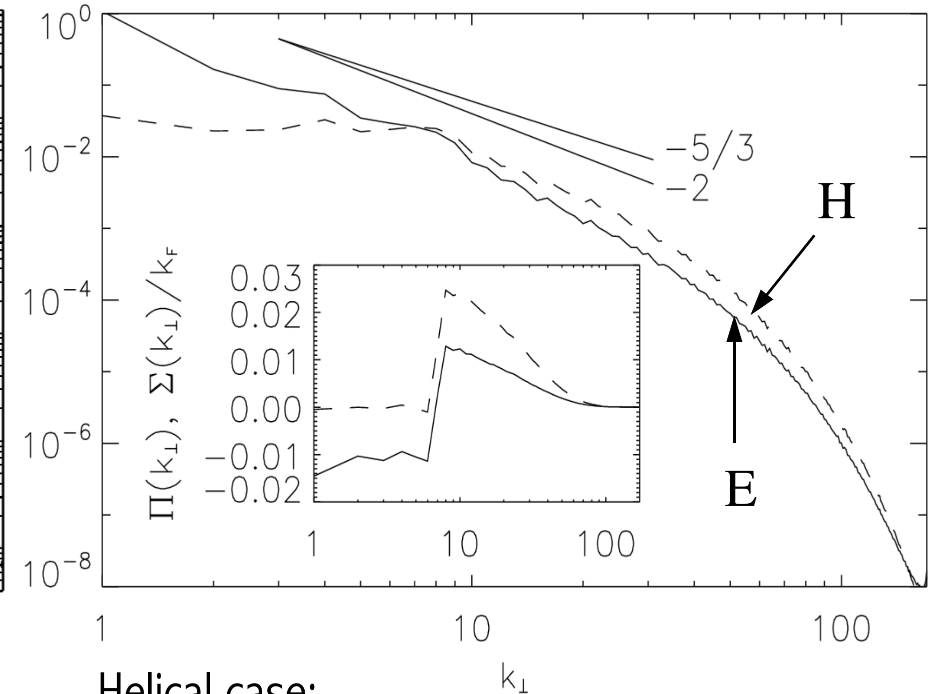




Enstrophy

Energy spectrum in rotating flows





Non-helical case:

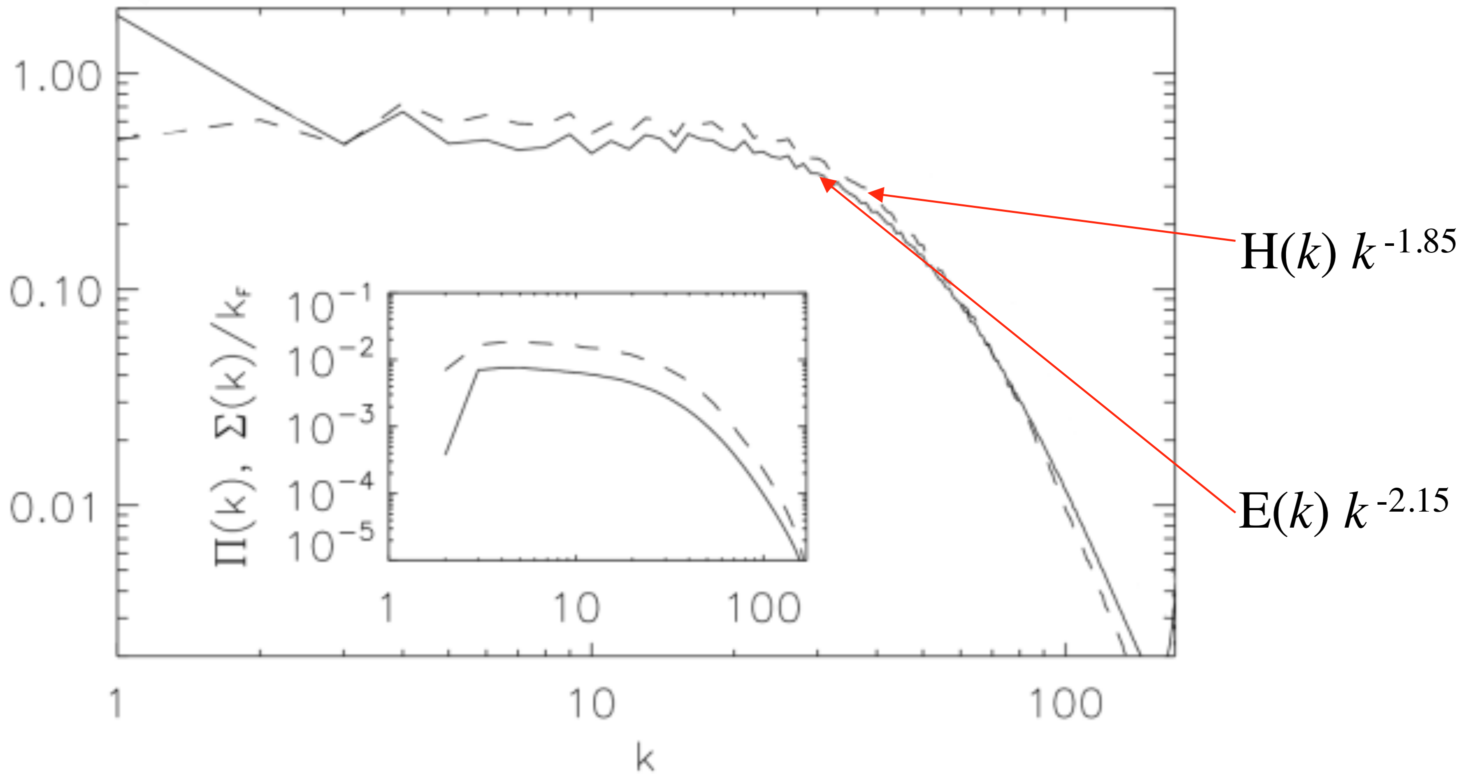
- An inverse cascade of energy develops for small *Ro*.
- The flow becomes anisotropic.
- The spectrum goes towards $k_\perp^{-2}$.

Helical case:

- Inverse cascade of energy and direct cascade of helicity.
- The direct energy flux is sub-dominant to the helicity flux.
- The energy spectrum becomes steeper than $k_\perp^{-2}$.

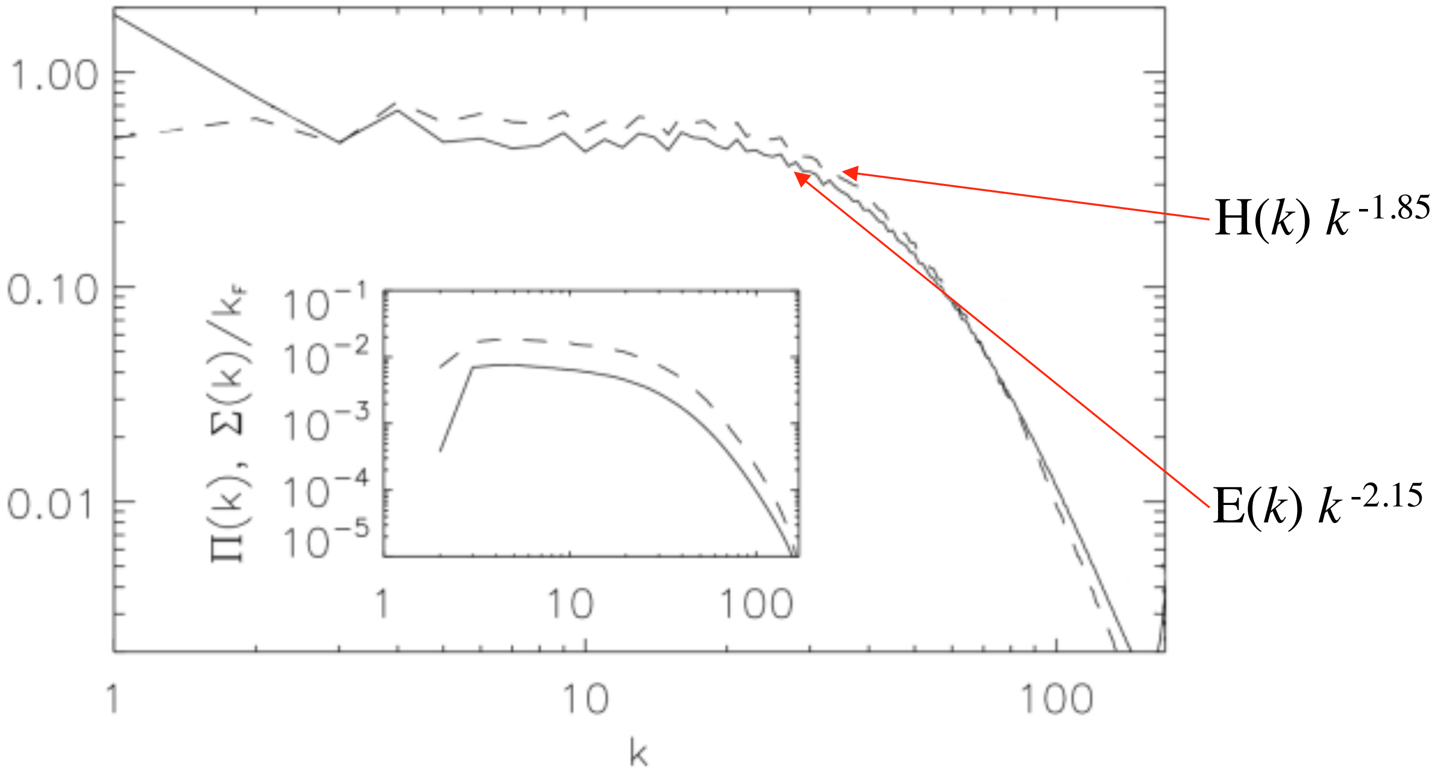
Mininni, Alexakis & Pouquet, PoF **21**, 015108; Mininni & Pouquet, PRE **79**, 026304 (2009)

The helicity cascade



Mininni & Pouquet, PRE **79**, 026304 (2009), Phys. Fluids **22**, 035105 (2010), JFM **699**, 263 (2012)

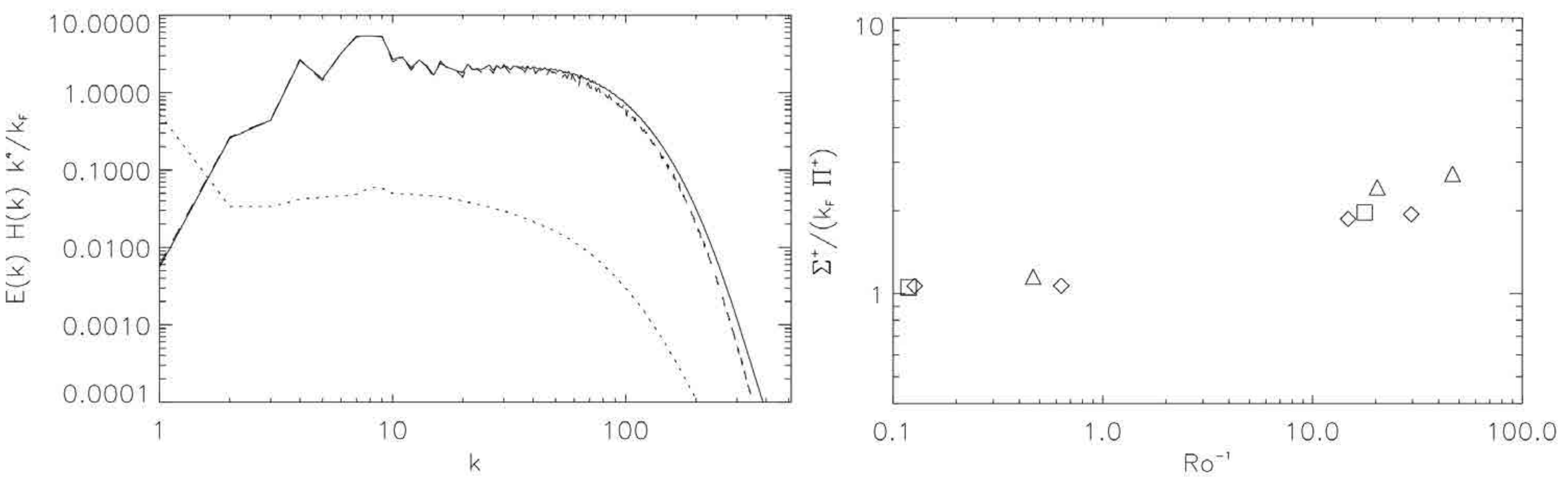
Helical rotating turbulence



- With rotation, energy goes towards large scales and helicity dominates the direct cascade: the helicity flux is constant $\delta \sim h_l \tau_\Omega / \tau_l^2 \sim h_l u_l^2/(l_\perp^2 \Omega)$, and $h_l \sim l_\perp^2/u_l^2$.
- If $E(k_\perp) \sim k_\perp^{-n}$, $H(k_\perp) \sim k_\perp^{-4+n}$ or $E(k_\perp)H(k_\perp) \sim k_\perp^{-4}$
- From Schwarz, $n \leq 2.5$ (the equality corresponds to maximum helicity).

Mininni & Pouquet, PRE **79**, 026304 (2009), Phys. Fluids **22**, 035105 (2010), JFM **699**, 263 (2012)

The k^{-4} spectrum and the direct helicity flux

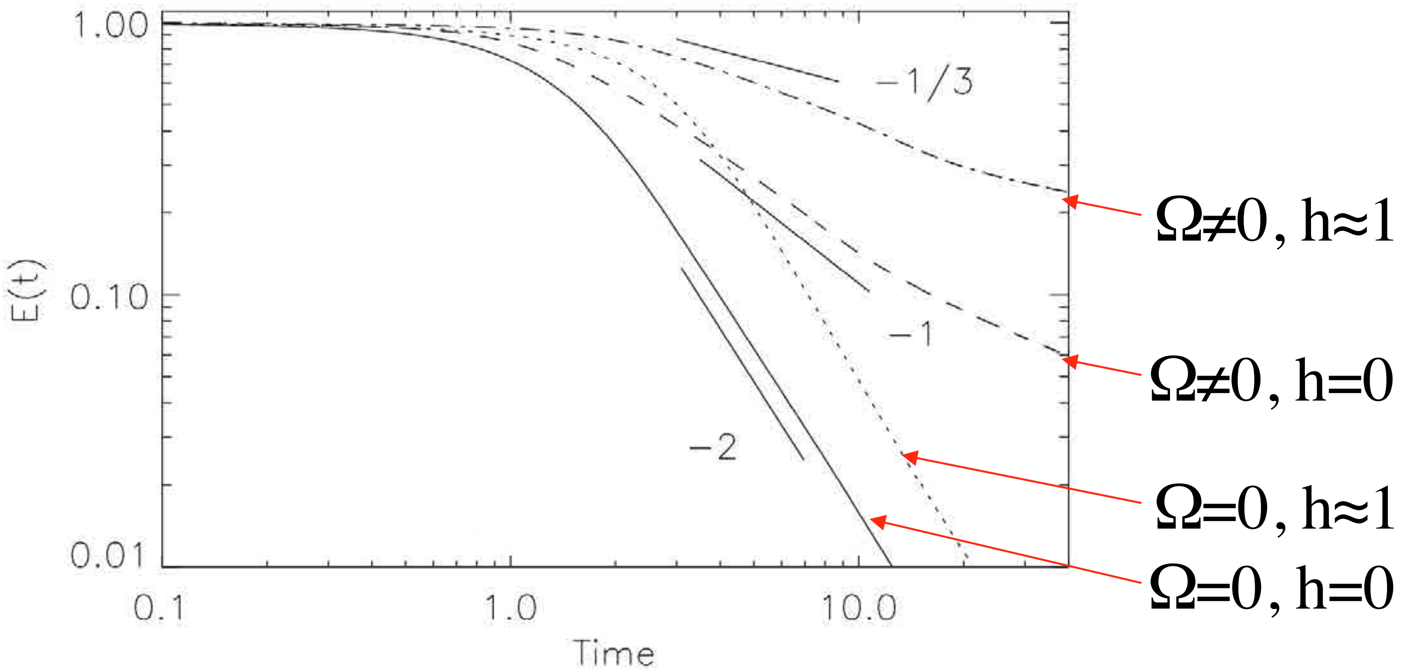


- The product of the energy and helicity spectra follow a $\sim k_\perp^{-4}$ law in several runs with rotation and helicity.
- The amount of helicity flux that goes towards small scales (normalized by the direct energy flux) increases with decreasing Rossby number, indicating the dominance of a direct cascade of helicity. Baerenzung et al., JAS (2011).
- The "$n+m = 4$" rule has been shown recently to be exact for rotating turbulence in the weak turbulence regime (Galtier 2014).

Are there any implications?

- Does the presence of helicity affect the decay of turbulence? Does it affect the lifetime of structures?
- Note different decay laws have been measured in simulations and experiments. Morize, Moisy, and Rabaud 2005; Morize and Moisy 2006, van Bokhoven et al. 2008, Davidson 2010.
- Does helicity affect the turbulent transport and diffusion of contaminants?

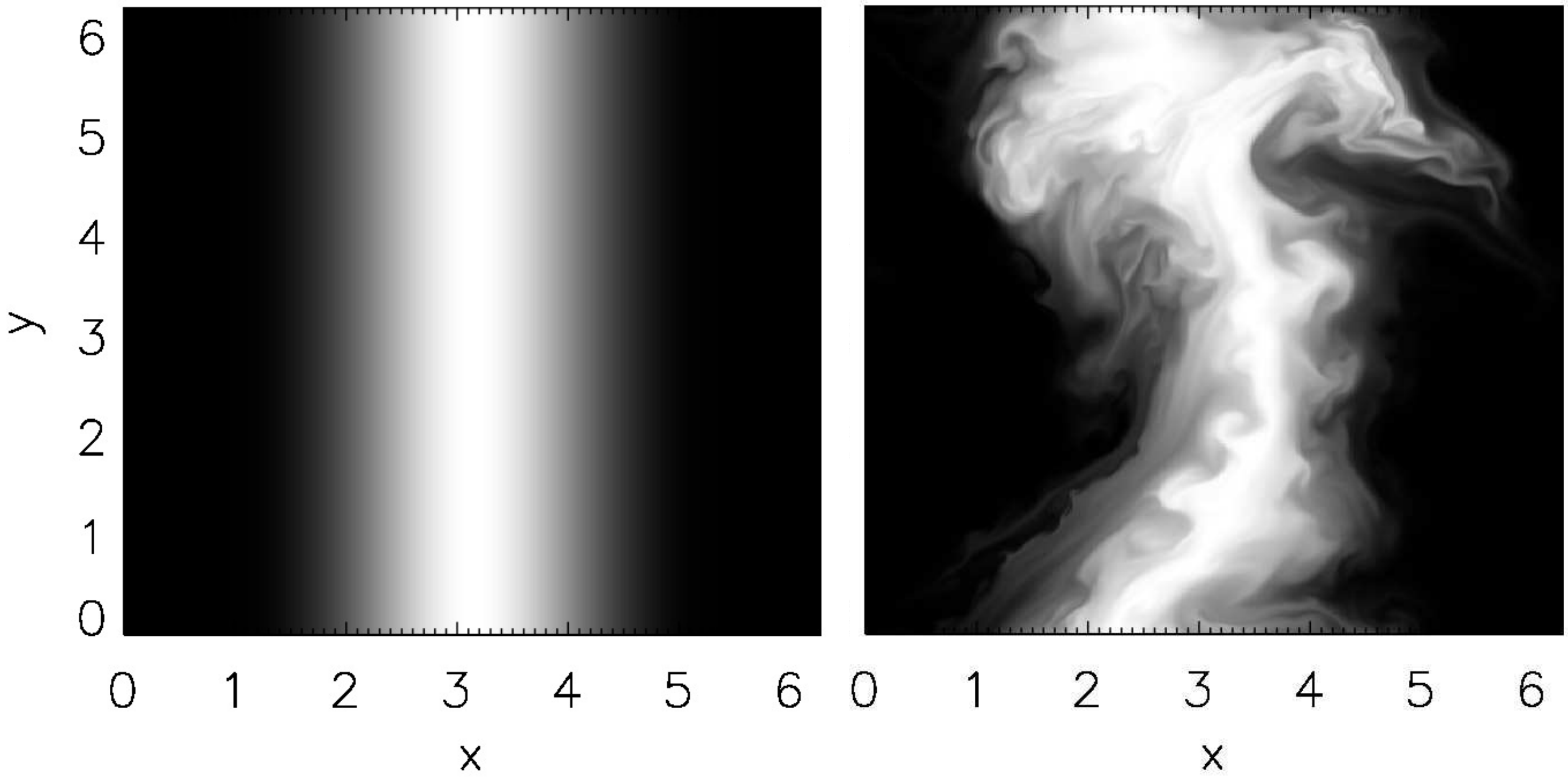
Freely decaying flows



- Simulations of bounded freely decaying turbulence, with and without rotation/helicity.
- Without rotation, helicity plays no role in the decay, except for a delay of the beginning of the self-similar regime
- With rotation, the helical flow decays slower.
- The decay laws can be correctly predicted taking into account the presence of helicity.

Teitelbaum & Mininni, PRL **103**, 014501 (2009)

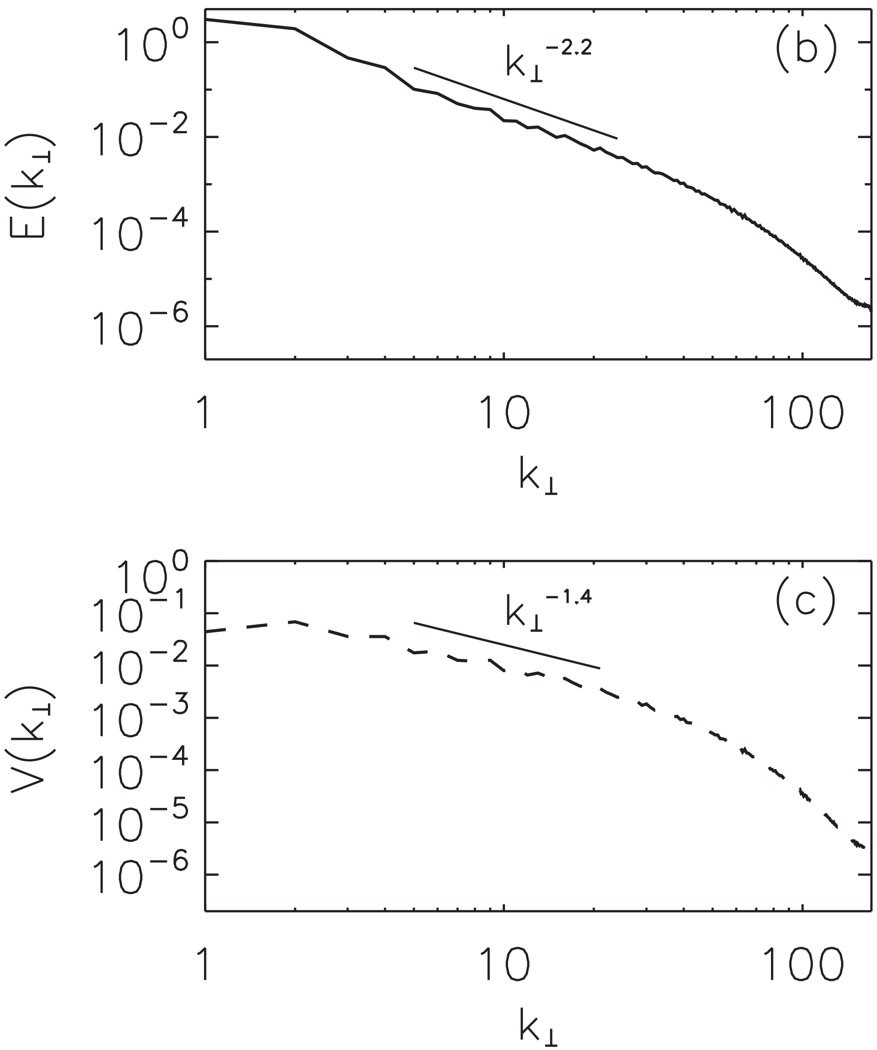
Transport and mixing

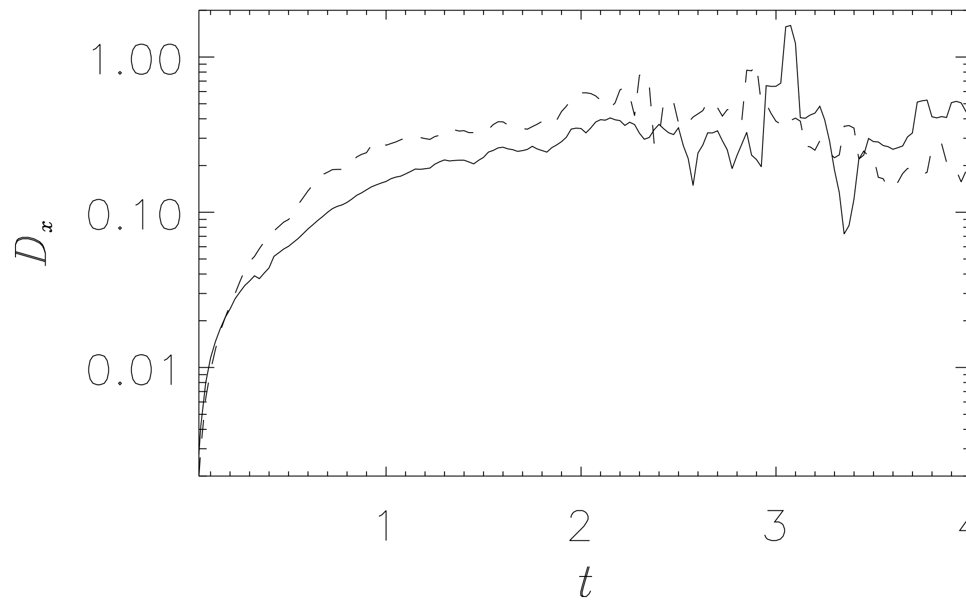


- Horizontal turbulent diffusion of a passive scalar is smaller in rotating helical flows than in rotating non-helical flows.

Rodriguez Imazio & Mininni, PRE **87**, 023018 (2013)

Transport and mixing





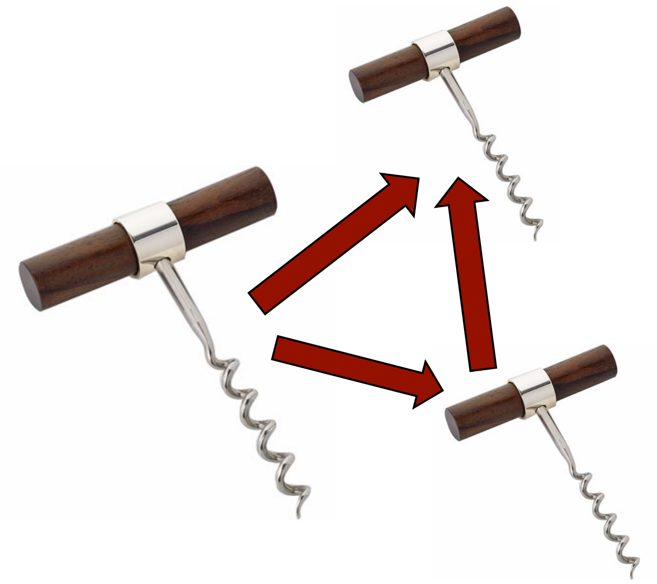
Rodriguez Imazio & Mininni, Phys. Scripta (2013), PRE (submitted)

Regularity

- From the momentum equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla P + \nu \nabla^2 \mathbf{v} + \mathbf{F}$$

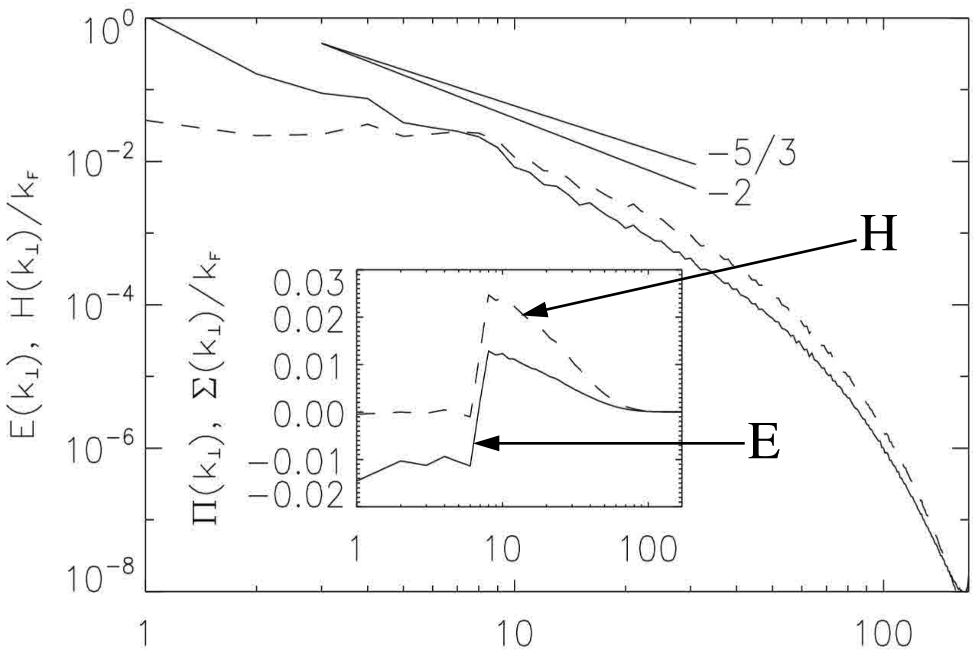
$$\Rightarrow \frac{dE(k)}{dt} = -\sum_{p,q} \int \mathbf{v}_k \cdot \left[(\mathbf{v}_p \cdot \nabla) \mathbf{v}_q \right] d^3x - 2\nu Z(k) + \varepsilon(k)$$



Biferale & Titi (2013)

Regularity

- A helical-decimated version of 3D Navier-Stokes displays an inverse cascade of energy, with a direct cascade of helicity.
- The system also has regular solutions (i.e., no singularity).



Mininni & Pouquet, PRE **79**, 026304 (2009)

Biferale & Titi (2013)

Rotating and stratified flows

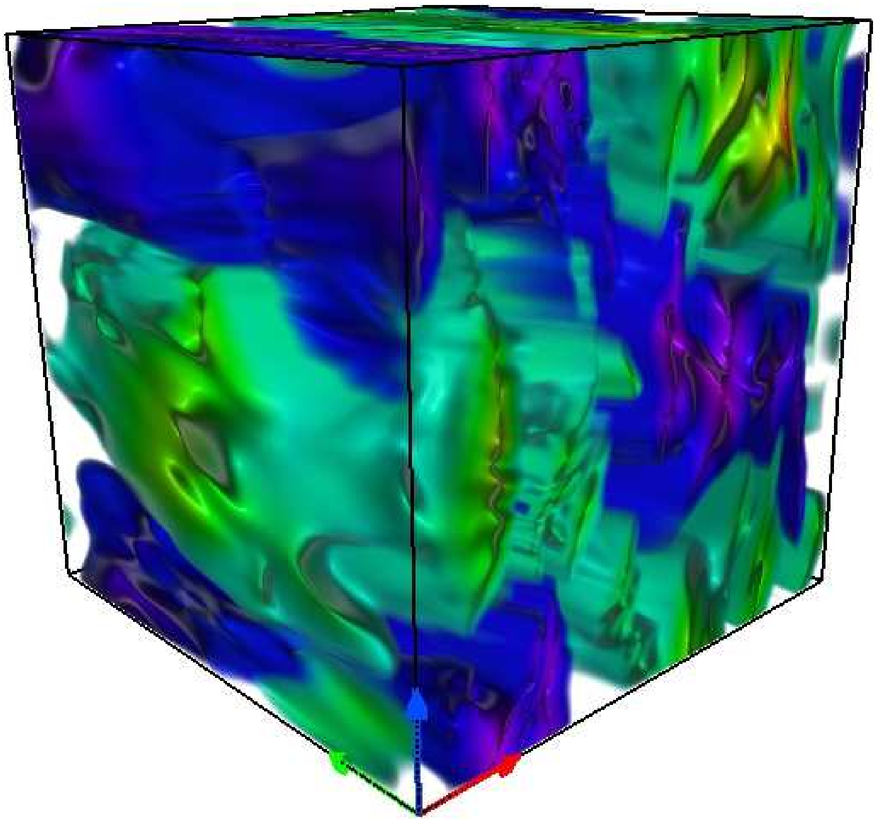
- Can we generate large-scale helicity in a "realistic" way?
- Momentum equation plus (potential) temperature equation

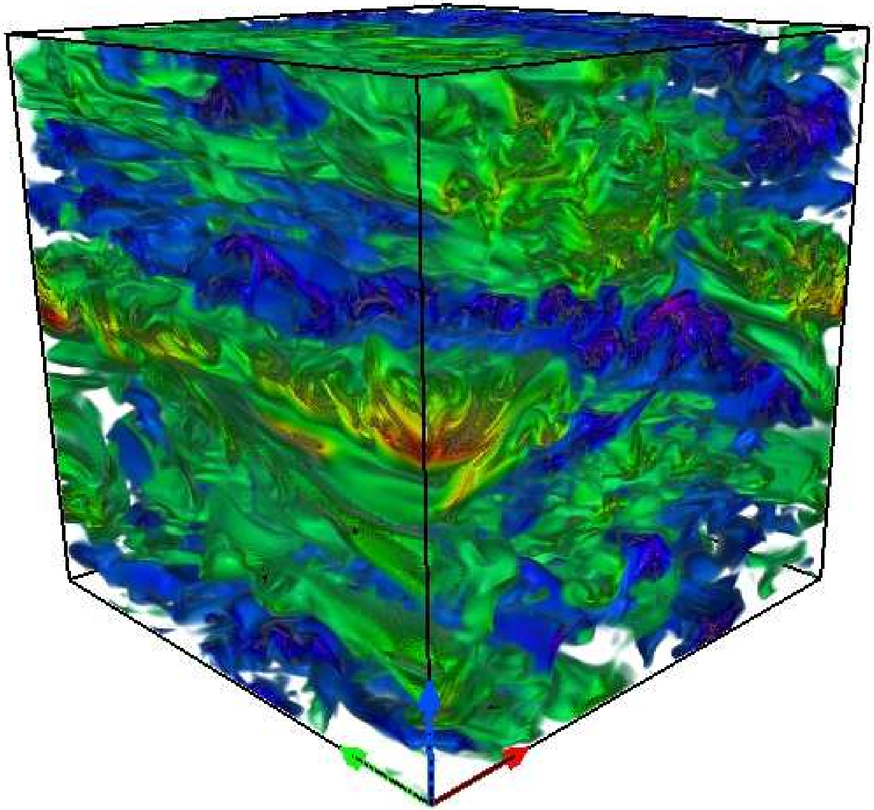
$$
\begin{aligned}
\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} &= -\nabla P - N\theta \boldsymbol{e}_z - 2\Omega \boldsymbol{e}_z \times \mathbf{u} \\
\partial_t \theta + \mathbf{u}\cdot\nabla\theta - \kappa\Delta\theta &= Nw, \\
\nabla\cdot\mathbf{u} &= 0 .
\end{aligned}
$$

- Buoyancy now acts as a restitutive force, allowing for internal gravity waves.
- In the ideal case, helicity is not conserved anymore, but total energy (kinetic plus potential) is.
- Froude, Rossby, and Reynolds numbers

$$
Fr = \frac{u_0}{NL_0}\ , \ Ro = \frac{u_0}{fL_0}\ , \ Re = \frac{u_0 L_0}{\nu}
$$

Rotating and stratified flows





Helicity in rotating and stratified flows

- From the momentum equation

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} = -\nabla P - N\theta \boldsymbol{e}_z - 2\Omega \boldsymbol{e}_z \times \mathbf{u}$$

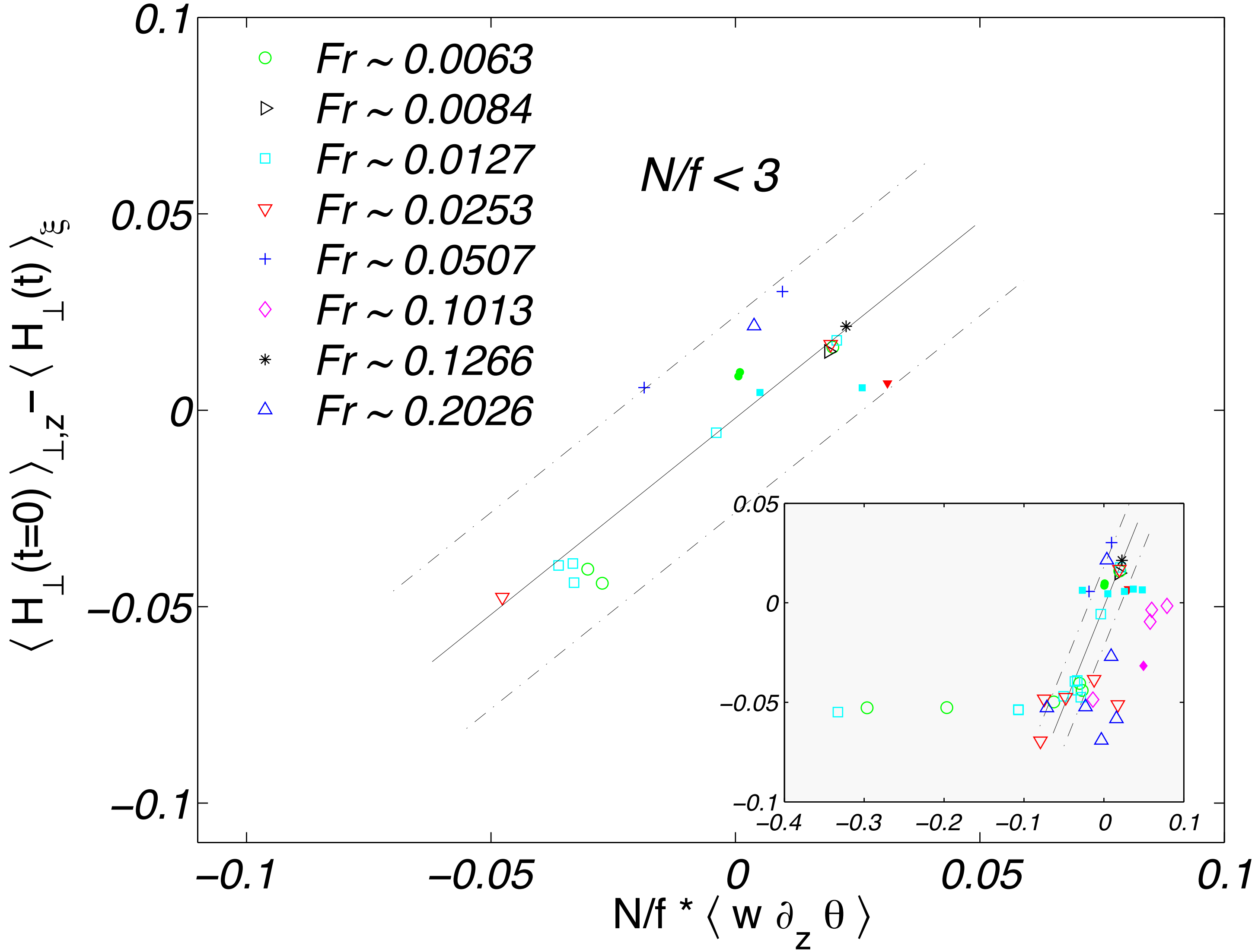
the equation for the time evolution of the helicity is

$$\frac{dH}{dt} = -2N\left<\theta\ \omega_z\right> - 2\nu Z_H, \quad Z_H = \left<\omega\cdot\nabla\times\omega\right>$$

At large scales, the following balance can be expected from pressure, buoyancy and Coriolis forces

$$\left<H_\perp\right>_{\perp,z} = -\frac{N}{f}\left<w\frac{\partial\theta}{\partial z}\right>_{\perp,z}$$

Generation of helicity



Marino et al., PRE **87**, 033016 (2013). See also Levina 2013 and Tur & Yanovsky 2013

Summary

- We can distinguish waves and eddies and quantify their strength in time and space resolved numerical simulations.
- The role of helicity in isotropic and homogeneous turbulence is unclear. Spectral studies are inconclusive as helicity is not positive definite.
- In the rotating case, two different spectra seem to arise, depending on the helicity content of the flow.
- The presence of helicity also affects the decay of rotating turbulence, and the transport and diffusion of passive scalars.
- Helicity is injected artificially in these simulations. However, the interplay between rotation and stratification can spontaneously create helicity in a flow.
- Once helicity is created, it affects the evolution of the flow, even in the absence of rotation.
- We are now extending these studies to purely stratified flows.